A Polynomial Algorithm for Subisomorphism of Open Plane Graphs

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A Polynomial Algorithm for Subisomorphism of Holey Plane Graphs

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Abstract

We address the problem of searching for a pattern in a plane graph, that is, a planar drawing of a planar graph. We define plane subgraph isomorphism and give a polynomial algorithm for this problem. We show that this algorithm may be used even when the pattern graph has holes.

1. Introduction

Many applications involve mining graphs in order to discover frequent connected subgraphs. If this problem may be solved in output-polynomial time for some specific classes of graphs such as trees (Chi et al., 2005), tenuous outerplanar graphs (Horvath et al., 2006), or bounded treewidth graphs (Horvath & Ramon, 2008), it remains challenging in the general case. This mainly comes from the fact that subgraph isomorphism is $\mathcal{NP}$-complete in the general case.

In this paper, we focus on plane graphs, i.e., planar graphs that are embedded in planes. Indeed, when graphs model objects defined on planes, such as images, one may consider the planar embedding of the graph. In (Damiand et al., 2009), we have defined and studied the plane (sub)graph isomorphism problem. We have shown that it can be solved in quadratic time whenever the pattern graph is *compact*, that is, the pattern graph may be obtained by iteratively removing nodes and edges that are incident to the unbounded face.

However, compact plane graphs are somehow restricted, because they do not have any hole. Thus, it would be impossible to use compact plane graphs to

Figure 1. Two cups that should be modelled by holey graphs in order to remove the background which is visible through the handle.

model and search for a cup with a handle for instance (see Fig. 1). Indeed, the background of the cup, visible through the handle, would be integrated to the modelling graph, so that the cup could not be searched independently of the background.

In this paper, we extend Damiand et al.(2009)'s algorithm to solve the sub-isomorphism problem for plane graphs with holes.

2. Plane Graphs

A *graph* is a pair $G = (V, E)$ where V is a set of *vertices* and E is a set of *edges*. Below, all the graphs are supposed connected, that is, every pair of vertices is linked by a sequence of edges.

A *planar embedding* of a graph G is an injective mapping ϕ that assigns 2D points to vertices, and 2D curves to edges. G is *planar* if an embedding exists such that no two embedded edges intersect, except at their endpoints. A theorem by Fàry (1948) states that given a non-crossing representation of a planar graph, it is always possible to move the vertices so that the edges are drawn with straight-line segments. Hence, we only consider planar embeddings such that embedded edges are straight-line segments that are defined by the 2D embedding of their endpoint vertices.

Several embeddings may however exist for a graph. A *plane graph* is a triple $G = (V, E, \phi)$ such that (V, E) is
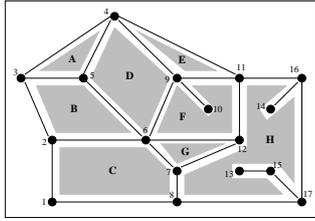
Figure 2. A face-connected plane graph G_1 with 8 bounded faces (labeled from A to H) and 1 unbounded (white) face.
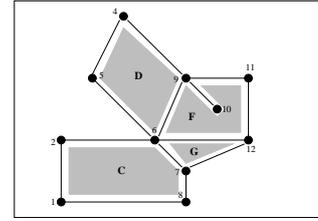

Figure 3. A face-connected plane graph G_2 which is a compact subgraph of G_1 (given in Fig. 2).

a planar graph and $\phi : V \to \mathbb{R}^2$ is an embedding of the vertices such that no two embedded edges intersect, except at their endpoints.

A plane graph is made of (bounded or unbounded) faces: when considering the planar embedding, the complement of the set of edges is a disjoint union of simply connected regions called *faces*. For instance, the plane graph of Fig. 2 is made of 9 faces: Faces A to H are bounded whereas the white face is unbounded. Let $faces(G)$ denote the set of faces defined by plane graph $G = (V, E, \phi)$. For each $f \in faces(G)$, we note $boundary(f)$ the sequence of vertices encountered when walking along the boundary of f, having f on the right hand side. This boundary is unique up to cyclic permutations.

We finally introduce *face-connectivity*, that is based on sequences of faces that share common edges. Formally, two faces $f, g \in faces(G)$ are *sewn* if there exists an edge $\{i, j\} \in E$ which belongs both to $boundary(f)$ and $boundary(g)$. Graph G is *face-connected* if for each $f, g \in faces(G)$, there exists a sequence of faces $f_1, f_2, \ldots f_n$ such that $f = f_1$, $g = f_n$, and faces f_i and f_{i+1} are pairwise sewn, for all $1 \le i \le n - 1$.

3. Compact Plane Subgraph Isomorphism and Combinatorial Maps

A compact plane subgraph isomorphism problem between a pattern plane graph $G_p = (V_p, E_p, \phi_p)$ and a target plane graph $G_t = (V_t, E_t, \phi_t)$ consists in deciding whether G_p is isomorphic to some subgraph of G_t which is obtained from G_t by iteratively removing nodes and edges that are adjacent to the unbounded face (see Fig. 3). More precisely, one should find a mapping $h : V_p \to V_t$ such that (i) h is injective, (ii) h preserves the edges, i.e., $\forall \{x, y\} \in E_p$, one has $\{h(x), h(y)\} \in E_t$, and (iii) h preserves the faces, i.e., for every face $f \in faces(G_p)$, there exists a face $g \in faces(G_t)$ such that for every edge $\{x, y\} \in$
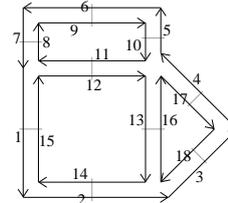

Figure 4. An example of combinatorial map. Darts are represented by numbered segments. Concerning functions β_1 and β_2, consider e.g. dart 11; one has $\beta_1(11) = 8$ and $\beta_2(11) = 12$.

$boundary(f)$, one has $\{h(x), h(y)\} \in boundary(f)$.

In (Damiand et al., 2009), we have proposed a polynomial algorithm for the plane subgraph isomorphism problem where the pattern graph G_p must be face-connected. This algorithm is derived from an algorithm that solves the sub-isomorphism problem for *combinatorial maps*. Combinatorial maps were introduced in the early 60's to efficiently implement plane graphs (Edmonds, 1960; Tutte, 1963). They describe the topological organisation of plane graphs by decomposing every edge $\{i, j\}$ into two darts $(i \to j)$ and $(j \to i)$, and by using two functions β_1 and β_2 which respectively define dart successions in face boundaries and face connectivity (see Fig. 4).

The algorithm proposed in (Damiand et al., 2009) for solving the submap isomorphism problem is based on the fact that, given any starting dart, the traversal of a combinatorial map (that is, the order in which darts are discovered) is unique, provided that one has fixed (1) the strategy used to memorize darts that were dicovered but not exploited yet (e.g., Last In First Out / LIFO), and (2) the order in which β_1 and β_2 are used to discover new darts (e.g., β_1 before β_2).

Hence, to determine if a pattern map M_p is subisomorphic to a target map M_t, we choose a starting dart d_p in M_p, and for every dart d_t of M_t, we perform a traversal of M_p starting from d_p, and a traversal of M_t starting from d_t. Each time a new dart is discov-

Figure 5. Finding a car in an image: The original image, coming from the Movi dataset (Luo et al., 2003), is on the left. The plane graph obtained after segmentation is on the middle. The car has been extracted and rotated on the right. It is found again in the original image using Damiand et al.(2009)'s algorithm.

ered in M_p, it is matched with the corresponding dart in M_t. At the end of the traversal, we check whether the matching that has been built actually defines a subisomorphism (i.e., the problem is solved) or not (i.e., we try with another starting dart in M_t).

Each traversal may be done in linear time w.r.t. the number of darts, and in the worst case one has to perform a traversal for every dart of M_t. Hence, this algorithm is quadratic w.r.t. the number of darts. Moreover, we have shown in (Damiand et al., 2009) that this algorithm for solving the submap isomorphism problem could be used to solve the compact plane subgraph isomorphism problem, provided that the pattern graph is face-connected. We have also shown that this algorithm could be used to find patterns in images modelled by plane graphs (see Fig. 5).

4. Plane Subgraph Isomorphism for Graphs with Holes

As pointed out in the introduction, when looking for a pattern in an image, one may want to remove some parts of the image (corresponding to the background). This could be done by modelling the pattern image with a holey plane graph, i.e., a plane graph such that some faces have been removed.

We define the plane subgraph isomorphism for graphs with holes as follows. Consider a pattern compact plane graph $G_p = (V_p, E_p, \phi_p)$, a set of required faces $\mathcal{F} \subseteq faces(G_p)$, and a target plane graph $G_t = (V_t, E_t, \phi_t)$. Let $V_p^{\mathcal{F}}$ denote the set of vertices that appear in $\mathcal{F}$ and $E_p^{\mathcal{F}}$, the corresponding set of edges. One should find a mapping $h : V_p^{\mathcal{F}} \to V_t$ such that (i) h is an injection, (ii) h preserves edges, i.e., $\forall \{x, y\} \in E_p^{\mathcal{F}}$, one has $\{h(x), h(y)\} \in E_t$, (iii) h preserves the faces

of $\mathcal{F}$, i.e., for every face $f \in \mathcal{F}$, there exists a face $g \in faces(G_t)$ such that for every edge $\{x, y\} \in boundary(f)$, one has $\{h(x), h(y)\} \in boundary(f)$.

For example, in Fig. 2, the graph obtained from G_1 by eliminating faces D and H (thus, setting $\mathcal{F} = \{A, B, C, E, F, G\}$) would be a plane graph with holes and a face-connected subgraph of graph G_1.

The algorithm for deciding the plane subgraph isomorphism problem for graphs with holes is derived from the submap isomorphism algorithm of (Damiand et al., 2009) as follows: in the traversal of the pattern graph, the faces that do not belong to the set of required faces $\mathcal{F}$ must not considered. Note that the set of required faces $\mathcal{F}$ has to be face-connected.

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