Approximate roots of a valuation and the Pierce-Birkhoff Conjecture

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APPROXIMATE ROOTS OF A VALUATION AND THE PIERCE–BIRKHOFF CONJECTURE

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Abstract

In this paper, we construct an object, called a system of approximate roots of a valuation, centered in a regular local ring, which describes the fine structure of the valuation (namely, its valuation ideals and the graded algebra). We apply this construction to valuations associated to a point of the real spectrum of a regular local ring A. We give two versions of the construction: the first, much simpler, in a special case (roughly speaking, that of rank 1 valuations), the second — in the case of complete regular local rings and valuations of arbitrary rank.

We then describe certain subsets $C \subset \text{Sper } A$ by explicit formulae in terms of approximate roots; we conjecture that these sets satisfy the Connectedness (respectively, Definable Connectedness) conjecture. Establishing this for a certain regular ring A would imply that A is a Pierce–Birkhoff ring (this means that the Pierce–Birkhoff conjecture holds in A).

Finally, we use these constructions and results to prove the Definable Connectedness conjecture (and hence *a fortiori* the Pierce–Birkhoff conjecture) in the special case when $\dim\ A = 2$.

Introduction

All the rings in this paper will be commutative with 1. Let R be a real closed field. Let $B = R[x_1, \dots, x_n]$. If A is a ring and $\mathfrak{p}$ a prime ideal of A, $\kappa(\mathfrak{p})$ will denote the residue field of $\mathfrak{p}$.

The Pierce–Birkhoff conjecture asserts that any piecewise-polynomial function

$$f : R^n \to R$$

can be expressed as a maximum of minima of a finite family of polynomials in n variables. We start by giving the precise statement of the conjecture as it was first stated by M. Henriksen and J. Isbell in the early nineteen sixties.

Definition 0.1.1 *A function $f : R^n \to R$ is said to be* ***piecewise polynomial*** *if R^n can be covered by a finite collection of closed semi-algebraic sets P_i such that for each i there exists a polynomial $f_i \in B$ satisfying $f|_{P_i} = f_i|_{P_i}$.*

Clearly, any piecewise polynomial function is continuous. Piecewise polynomial functions form a ring, containing B, which is denoted by $PW(B)$.

On the other hand, one can consider the (lattice-ordered) ring of all the functions obtained from B by iterating the operations of sup and inf. Since applying the operations of sup and inf to polynomials produces functions which are piecewise polynomial, this ring is contained in $PW(B)$ (the latter ring is closed under sup and inf). It is natural to ask whether the two rings coincide. The precise statement of the conjecture is:

Conjecture 0.1.2 (Pierce-Birkhoff) *If $f : R^n \to R$ is in $PW(B)$, then there exists a finite family of polynomials $g_{ij} \in B$ such that $f = \sup\limits_i \inf\limits_j \{g_{ij}\}$ (in other words, for all $x \in R^n$, $f(x) = \sup\limits_i \inf\limits_j \{g_{ij}(x)\}$).*

This paper represents the second step of our program for proving the Pierce–Birkhoff conjecture in its full generality. The starting point of this program is the abstract formulation of the conjecture in terms of the real spectrum of B and separating ideals proposed by J. Madden in 1989 [26], which we now recall, together with the relevant definitions. For a general introduction to real spectrum, we refer the reader to [7], Chapter 7, [3], Chapter II or [33], 4.1, page 81 and thereafter; see also "Bibliographical and historical comments" on p. 109 at the end of that chapter.

Let A be a ring. A point α in the real spectrum of A is, by definition, the data of a prime ideal $\mathfrak{p}$ of A, and a total ordering $\leq$ of the quotient ring $A/\mathfrak{p}$, or, equivalently, of the field of fractions of $A/\mathfrak{p}$. Another way of defining the point α is as a homomorphism from A to a real closed field, where two homomorphisms are identified if they have the same kernel $\mathfrak{p}$ and induce the same total ordering on $A/\mathfrak{p}$.

The ideal $\mathfrak{p}$ is called the support of α and denoted by $\mathfrak{p}_\alpha$, the quotient ring $A/\mathfrak{p}_\alpha$ by $A[\alpha]$, its field of fractions by $A(\alpha)$ and the real closure of $A(\alpha)$ by $k(\alpha)$. The total ordering of $A(\alpha)$ is denoted by $\leq_\alpha$. Sometimes we write $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$.

Definition 0.1.3 *The real spectrum of A, denoted by Sper A, is the collection of all pairs $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of A and $\leq_\alpha$ is a total ordering of $A/\mathfrak{p}_\alpha$.*

We use the following notation: for an element $f \in A$, $f(\alpha)$ stands for the natural image of f in $A[\alpha]$ and the inequality $f(\alpha) > 0$ really means $f(\alpha) >_\alpha 0$.
The real spectrum Sper A is endowed with two natural topologies. The first one, called the **spectral (or Harrison) topology**, has basic open sets of the form

$$U(f_1, \dots, f_k) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0\}$$

with $f_1, ..., f_k \in A$.

The second is the **constructible topology** whose basic open sets are of the form

$$V(f_1, \dots, f_k, g) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0, g(\alpha) = 0\},$$

where $f_1, ..., f_n, g \in A$. Boolean combinations of sets of the form $V(f_1, \dots, f_n, g)$ are called **constructible sets** of Sper A.

For more information about the real spectrum, see [7]; there is also a brief introduction to the real spectrum and its relevance to the Pierce–Birkhoff conjecture in the Introduction to [21].

Definition 0.1.4 *Let*

$$f : Sper\ A \to \coprod_{\alpha \in Sper\ A} A(\alpha)$$

be a map such that, for each $\alpha \in Sper\ A$, $f(\alpha) \in A(\alpha)$. We say that f is ***piecewise polynomial*** *(denoted by $f \in PW(A)$) if there exists a covering of Sper A by a finite family $(S_i)_{i\in I}$ of constructible sets, closed in the spectral topology, and a family $(f_i)_{i\in I}$, $f_i \in A$ such that, for each $\alpha \in S_i$, $f(\alpha) = f_i(\alpha)$.*

We call f_i a ***local representative*** *of f at α and denote it by f_α (f_α is, in general, not uniquely determined by f and α; this notation means that one such local representative has been chosen once and for all).*

Note that $PW(A)$ is naturally a lattice ring: it is equipped with the operations of maximum and minimum. Each element of A defines a piecewise polynomial function. In this way we get a natural injection $A \subset PW(A)$.

Definition 0.1.5 *A ring A is a Pierce-Birkhoff ring if, for each $f \in PW(A)$, there exist a finite collection of $f_{ij} \in A$ such that $f = \sup\limits_i \inf\limits_j f_{ij}$.*

In [26] Madden reduced the Pierce–Birkhoff conjecture to a purely local statement about separating ideals and the real spectrum. Namely, he gave the following definition:

Definition 0.1.6 *Let A be a ring. For $\alpha, \beta \in Sper\ A$, the* ***separating ideal*** *of α and β, denoted by $< \alpha, \beta >$, is the ideal of A generated by all the elements $f \in A$ which change sign between α and β, that is, all the f such that $f(\alpha) \geq 0$ and $f(\beta) \leq 0$.*

Definition 0.1.7 *A ring A is* ***locally Pierce-Birkhoff*** *at α, β if the following condition holds. Let f be a piecewise polynomial function, let $f_\alpha \in A$ be a local representative of f at α and $f_\beta \in A$ a local representative of f at β. Then $f_\alpha - f_\beta \in < \alpha, \beta >$.*

Theorem 0.1.8 *(Madden) A ring A is Pierce-Birkhoff if and only if it is locally Pierce-Birkhoff for all $\alpha, \beta \in Sper(A)$.*

Let α, β be points in Sper A.

Conjecture 0.1.9 (local Pierce-Birkhoff conjecture at α, β) *Let A be a regular ring and f a piecewise polynomial function. Let $f_\alpha \in A$ be a local representative of f at α and $f_\beta \in A$ a local representative of f at β. Then $f_\alpha - f_\beta \in < \alpha, \beta >$.*

There are known counterexamples in the case A is not regular (eg. $A = R[x,y]/(y^2 - x^3)$) and even with A normal.

Remark 0.1.10 *Assume that β is a specialization of α. Then*

(1) $< \alpha, \beta > = \mathfrak{p}_\beta$.

(2) $f_\alpha - f_\beta \in \mathfrak{p}_\beta$. Indeed, we may assume that $f_\alpha \neq f_\beta$, otherwise there is nothing to prove. Since $\beta \in \overline{\{\alpha\}}$, f_α is also a local representative of f at β. Hence $f_\alpha(\beta) - f_\beta(\beta) = 0$, so $f_\alpha - f_\beta \in \mathfrak{p}_\beta$.

Therefore, to prove that a ring A is Pierce-Birkhoff, it is sufficient to verify Definition 0.1.7 for all α, β such that neither of α, β is a specialization of the other.

In [21], we introduced

Conjecture 0.1.11 (the Connectedness conjecture) *Let A be a regular ring. Let*

$$\alpha,\beta\in \textit{Sper } A$$

and let $g_1,\dots,g_s$ be a finite collection of elements of $A\setminus<\alpha,\beta>$. Then there exists a connected set $C\subset$ Sper A such that $\alpha,\beta\in C$ and $C\cap\{g_i=0\}=\emptyset$ for $i\in\{1,\dots,s\}$ (in other words, α and β belong to the same connected component of the set Sper $A\setminus\{g_1\dots g_s=0\}$).

Definition 0.1.12 *A subset C of Sper(A) is said to be **definably connected** if it is not a union of two non-empty disjoint constructible subsets, relatively closed for the spectral topology.*

Conjecture 0.1.13 (Definable connectedness conjecture) *Let A be a regular ring. Let $\alpha,\beta\in$ Sper A and let $g_1,\dots,g_s$ be a finite collection of elements of A, not belonging to $<\alpha,\beta>$. Then there exists a definably connected set $C\subset$ Sper A such that $\alpha,\beta\in C$ and $C\cap\{g_i=0\}=\emptyset$ for $i\in\{1,\dots,s\}$ (in other words, α and β belong to the same definably connected component of the set Sper $A\setminus\{g_1\dots g_s=0\}$).*

In the earlier paper [21] we stated the Connectedness conjecture (in the special case $A=B$) and proved that it implies the Pierce–Birkhoff conjecture. Exactly the same proof applies verbatim to show that the Definable Connectedness conjecture implies the Pierce-Birkhoff conjecture for any ring A.

One advantage of the Connectedness conjecture is that it is a statement about A (respectively, about the polynomial ring if $A=B$) which makes no mention of piecewise polynomial functions.

Our problem is therefore one of constructing connected subsets of Sper A having certain properties.

Terminology: If A is an integral domain, the phrase "valuation of A" will mean "a valuation of the field of fractions of A, non-negative on A". Also, we will sometimes commit the following abuse of notation. Given a ring A, a prime ideal $\mathfrak{p}\subset A$, a valuation ν of $\frac{A}{\mathfrak{p}}$ and an element $x\in A$, we will write $\nu(x)$ instead of $\nu(x \mod \mathfrak{p})$, with the usual convention that $\nu(0)=\infty$, which is taken to be greater than any element of the value group.

Given any ordered domain D, let $\bar{D}$ denote the convex hull of D in its field of fractions $D_{(0)}$:

$$\bar{D}:=\left\{f\in D_{(0)}\;\middle|\; d>|f| \text{ for some } d\in D\right\}.$$

The ring $\bar{D}$ is a valuation ring, since for any element $f\in D_{(0)}$, either $f\in\bar{D}$ or $f^{-1}\in\bar{D}$. For a point $\alpha\in$ Sper A, we define $R_\alpha:=\overline{A[\alpha]}$. In this way, to every point $\alpha\in$ Sper A we can canonically associate a valuation ν_α of $A(\alpha)$, determined by the valuation ring R_α. The maximal ideal of R_α is $M_\alpha=\left\{x\;\in A(\alpha)\;\middle|\;|x|<\frac{1}{|z|},\ \forall z\in A[\alpha]\setminus\{0\}\right\}$; its residue field k_α comes equipped with a total ordering, induced by $\le_\alpha$.

Let $U(R_\alpha)$ denote the multiplicative group of units of R_α and Γ_α the value group of ν_α. Recall that

$$\Gamma_\alpha\cong\frac{A(\alpha)\setminus\{0\}}{U(R_\alpha)}$$

and that the valuation ν_α can be identified with the natural homomorphism

$$A(\alpha)\setminus\{0\}\to\frac{A(\alpha)\setminus\{0\}}{U(R_\alpha)}.$$

By definition, we have a natural ring homomorphism

$$A\to R_\alpha \tag{1}$$

whose kernel is $\mathfrak{p}_\alpha$.

Conversely, the point α can be reconstructed from the ring R_α by specifying a certain number of sign conditions (finitely many conditions when A is noetherian) ([5], [17], [7] 10.1.10, p. 217).

The valuation ν_α has the following properties:
(1) $\nu_\alpha(A[\alpha]) \geq 0$
(2) If A is an R-algebra then for any positive elements $y, z \in A(\alpha)$,

$$\nu_\alpha(y) < \nu_\alpha(z) \implies y > Nz, \ \forall N \in R. \tag{2}$$

A ν_α-ideal of A is the preimage in A of an ideal of R_α. See [32] or [3], §II.3 for more information on this subject.

As pointed out in [21], the points of Sper A admit the following geometric interpretation (see also [10], [15], [32], p. 89 and [34] for the construction and properties of generalized power series rings and fields).

Definition 0.1.14 *Let k be a field and Γ an ordered abelian group. The generalized formal power series field $k\left(\left(t^\Gamma\right)\right)$ is the field formed by elements of the form $\sum\limits_{\gamma\in\Gamma} a_\gamma t^\gamma$, $a_\gamma \in k$ such that the set $\{\gamma \mid a_\gamma \neq 0\}$ is well ordered.*

The field $k\left(\left(t^\Gamma\right)\right)$ is equipped with the natural t-adic valuation v with values in Γ, defined by $v(f) = \inf\{\gamma \mid a_\gamma \neq 0\}$ for $f = \sum\limits_\gamma a_\gamma t^\gamma \in k\left(\left(t^\Gamma\right)\right)$. The valuation ring of this valuation is the ring $k\left[\left[t^\Gamma\right]\right]$ formed by all the elements of $k\left(\left(t^\Gamma\right)\right)$ of the form $\sum\limits_{\gamma\in\Gamma_+} a_\gamma t^\gamma$. Specifying a total ordering on k and $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$ sign conditions defines a total ordering on $k\left(\left(t^\Gamma\right)\right)$. In this ordering $|t|$ is smaller than any positive element of k. For example, if $t^\gamma > 0$ for all $\gamma \in \Gamma$ then $f > 0$ if and only if $a_{v(f)} > 0$.

For an ordered field k, let $\bar{k}$ denote the real closure of k. The following result is a variation on a theorem of Kaplansky ([15], [16]) for valued fields equipped with a total ordering.

Theorem 0.1.15 ([34], p. 62, Satz 21) *Let K be a real valued field, with residue field k and value group Γ. There exists an injection $K \hookrightarrow \bar{k}\left(\left(t^\Gamma\right)\right)$ of real valued fields.*

Let $\alpha \in$ Sper A. In view of (1) and the Remark above, specifying a point $\alpha \in$ Sper A is equivalent to specifying a total order of k_α, a morphism

$$A[\alpha] \to \bar{k}_\alpha\left[\left[t^{\Gamma_\alpha}\right]\right] \tag{3}$$

and $\dim_{\mathbb{F}_2}(\Gamma_\alpha/2\Gamma_\alpha)$ sign conditions.

We may pass to Zariski spectra to obtain morphisms

$$\text{Spec }\left(\bar{k}_\alpha\left[\left[t^{\Gamma_\alpha}\right]\right]\right) \to \text{Spec } A[\alpha] \to \text{Spec } A,$$

induced by the ring homomorphism (3) and the natural surjective homomorphism $A \twoheadrightarrow A[\alpha]$, respectively.

In particular, if $\Gamma_\alpha = \mathbb{Z}$, we obtain a **formal curve** in Spec A (an analytic curve if the series are convergent). This motivates the following definition:

Definition 0.1.16 *Let k be an ordered field. A k-**curvette** on Sper(A) is a morphism of the form*

$$\alpha : A \to k\left[\left[t^\Gamma\right]\right],$$

*where Γ is an ordered group. A k-**semi-curvette** is a k-curvette α together with a choice of the sign data sgn x_1,..., sgn x_r, where $x_1, ..., x_r$ are elements of A whose t-adic values induce an $\mathbb{F}_2$-basis of $\Gamma/2\Gamma$.*

We have thus explained how to associate to a point α of Sper A a $\bar{k}_\alpha$-semi-curvette. Conversely, given an ordered field k, a k-semi-curvette α determines a prime ideal $\mathfrak{p}_\alpha$ (the ideal of all the elements of A which vanish identically on α) and a total ordering on $A/\mathfrak{p}_\alpha$ induced by the ordering of the ring $k\left[\left[t^\Gamma\right]\right]$ of formal power series.

Below, we will often describe points in the real spectrum by specifying the corresponding semi-curvettes.

Let ν be a valuation centered in a regular local ring A (see §1.1), let $\Phi = \nu(A \setminus \{0\})$; Φ is a well-ordered set. For an ordinal $\lambda < \Phi$, let γ_λ be the element of Φ corresponding to λ.

Definition 0.1.17 *A system of approximate roots of ν is a well-ordered set of elements*

$$\mathbf{Q} = \{Q_i\}_{i\in\Lambda} \subset A,$$

satisfying the following condition: for every ν-ideal I in A, we have

$$I = \left\{ \prod_j Q_j^{\gamma_j} \;\middle|\; \sum_j \gamma_j \nu(Q_j) \geq \nu(I) \right\} A; \tag{4}$$

furthermore, we require the set $\mathbf{Q}$ to be minimal in the sense of inclusion among those satisfying (4).

A system of approximate roots of ν up to γ_λ is a well-ordered set of elements of A satisfying (4) for all the ν-ideals I such that $\nu(I) \leq \gamma_\lambda$.

The main results of this paper are:

1. Given a regular local ring $(A, \mathfrak{m}, k)$, a valuation ν centered at A, as above, and an element $\gamma_\lambda \in \Phi$ such that the ν-ideal determined by γ_λ is $\mathfrak{m}$-primary, we construct a system of approximate roots up to γ_λ.
2. We construct a system of approximate roots for A and ν under the assumption that A is $\mathfrak{m}$ -adically complete.
3. In the situation of the Connectedness (or Definable Connectedness) conjecure we describe certain subsets $C \subset$ Sper A by explicit formulae in terms of approximate roots; we conjecture that these sets satisfy the Connectedness (respectively, Definable Connectedness) conjecture.
4. In the special case dim $A = 2$, we use the above results and constructions to prove the Definable Connectedness conjecture (and hence *a fortiori* the Pierce–Birkhoff conjecture). We also prove the Connectedness conjecture in dimension 2, provided the ring A is excellent.

The paper is organized as follows. Sections 1.1 to 1.5 are purely valuation-theoretic; sections 1.2 and 1.4 are devoted to the construction of a system of approximate roots.

The approximate roots Q_i are constructed recursively in i. Roughly speaking, Q_{i+1} is the lifting to A of the minimal polynomial equation satisfied by $\mathrm{in}_\nu Q_i$ over $k\left[\{\mathrm{in}_\nu Q_j\}_{j<i}\right]$ in $\mathrm{gr}_\nu A$. In sections 1.1 to 1.5, we prove that such systems of approximate roots exist in two situations: first, for any $\mathfrak{m}$-primary ν-ideal J there exists a system of approximate roots up to $\nu(J)$; secondly, there exists a system of approximate roots whenever A is $\mathfrak{m}$-adically complete.

Once these valuation-theoretic tools are developed, we continue with the program announced in [21] for proving the Pierce-Birkhoff conjecture. We place ourselves in the situation of Conjectures 0.1.11 and 0.1.13. In §2.1 we describe the separating ideal $<\alpha,\beta>$ by describing monomials in the approximate roots (common to the valuations ν_α and ν_β) which generate it. In section 2.2, we give an explicit description of a set $C \subset \mathrm{Sper}A \setminus \{g_1 \dots g_s = 0\}$, containing α and β, which we conjecture to be connected. The set C is described in terms of a finite family of approximate roots, common to the valuations ν_α and ν_β.

Finally, we prove the Definable connectedness conjecture and hence the Pierce-Birkhoff conjecture for an arbitrary regular 2-dimensional local ring A; we also prove Conjecture 0.1.11 assuming that A is excellent which provides a second proof of the Pierce-Birkhoff conjecture in the case of excellent rings. The outline of the proof of the two conjectures is as follows. First, we use a sequence of point blowings up and Zariski's theory of complete ideals (recalled and refined in §3.1) to transform the set C into a set U of a very simple form, which informally we call a quadrant. Namely, U is the set of all the points δ of Sper A' (where A' is a regular two-dimensional local ring obtained after a sequence of blowings up with regular system of parameters x', y'), centered at the origin, which induce a specified total order on k and which satify the sign conditions $x'(\delta) > 0$, $y'(\delta) > 0$ (resp. $x'(\delta) > 0$). This is accomplished in §3.3.2.

In the special case when A' is essentially of finite type over a real closed field the connectedness of U is well known and follows easily from the results of [7] (which allow to reduce connectedness of U to that of a quadrant in the usual Euclidean plane). However, for more general regular rings this result seems to us to be new and non-trivial.

In §3.4, we use results from [3] to reduce the connectedness of U to that of a quadrant in the usual Euclidean space, assuming the ring A is excellent. This completes the proof of the connectedness conjecture for excellent regular 2-dimensional rings. In §3.5 we prove the definable connectedness of U, without any excellence assumptions, by using a new notion of a graph, associated to a sequence of point blowings-up of a real surface.

Our proof is based on Madden's unpublished preprint [27]. As well, we would like to acknowledge a recent paper by S. Wagner [44] which gives a proof of the Definable Connectedness and the Pierce–Birkhoff conjecture in the case of smooth 2-dimensional algebras of finite type over real closed fields.

The overall structure of our proof is similar to that of [27] and [44], with the following differences:

1. Here, we have tried to present a proof which should provide a pattern for a general proof of the conjecture, that is, have a hope of generalizing to higher dimensions. In particular, we went to great lengths to phrase everything in terms of approximate roots rather than work directly with connected sets as in [27] and [44].
2. We make no assumptions on the real closedness of the residue field of A which introduces certain extra complications.
3. Because we work with arbitrary regular two-dimensional rings, we have to overcome a serious difficulty: proving that the "quadrant" U, defined above, is connected. This is well known for algebras of finite type over a real closed field (see, for example, [7]) but as far as we can tell, for general rings this result is new and non-trivial. Its proof occupies most of section 3.5.

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Part 1. Valuations and approximate roots.

1.1 Generalities on valuations.

In this section we review some basic facts of valuation theory.

Let A be a noetherian ring and $\nu : A \to \Gamma \cup \{\infty\}$ a valuation centered at a prime ideal of A. Let $\Phi = \nu(A \setminus \{0\}) \subset \Gamma$.

For each $\gamma \in \Phi$, consider the ideals

$$\begin{aligned} P_\gamma &= \{x \in A \mid \nu(x) \geq \gamma\} \\ P_{\gamma+} &= \{x \in A \mid \nu(x) > \gamma\}. \end{aligned} \tag{5}$$

P_γ is called the **ν-ideal** of A of value γ.

Remark 1.1.1 *It is easy to see that, as A is noetherian, $\nu(A)$ is well-ordered.*

Notation. If I is an ideal of A and ν a valuation of A, $\nu(I)$ will denote $\min\{\nu(x) \mid x \in I\}$.

We now define certain natural graded algebras associated to a valuation. Let A, ν and Φ be as above. For $\gamma \in \Phi$, let P_γ and $P_{\gamma+}$ be as in (5). We define

$$\mathrm{gr}_\nu A = \bigoplus_{\gamma \in \Phi} \frac{P_\gamma}{P_{\gamma+}}.$$

The algebra $\mathrm{gr}_\nu(A)$ is an integral domain. For any element $f \in A$ with $\nu(f) = \gamma$, we may consider the natural image of f in $\frac{P_\gamma}{P_{\gamma+}} \subset \mathrm{gr}_\nu(A)$. This image is a homogeneous element of $\mathrm{gr}_\nu(A)$ of degree γ, which we denote by $\mathrm{in}_\nu f$. The grading induces an obvious valuation on $\mathrm{gr}_\nu(A)$ with values in Φ; this valuation will be denoted by ord.

We end this section with the notion of a *monomial* valuation. Let $(A, \mathfrak{m}, k)$ be a regular local ring, and $\mathbf{u} = (u_1, \dots, u_n)$ a regular system of parameters of A. Let Φ be an ordered semigroup and let $\beta_1, \dots, \beta_n$ be strictly positive elements of Φ. Let Φ_* denote the ordered semigroup, contained in Φ, consisting of all the $\mathbb{N}_0$-linear combinations of $\beta_1, \dots, \beta_n$. For $\gamma \in \Phi_*$, let I_γ denote the ideal of A, generated by all the monomials u^α such that $\sum\limits_{j=1}^{n} \alpha_j \beta_j \geq \gamma$ (we take $I_0 = A$). Let x be a non-zero element of A. Let $\Phi_x = \{\gamma \in \Phi_* \mid x \in I_\gamma\}$. Then it is not difficult to prove that the set Φ_x contains a maximal element and there exists a unique valuation ν, centered at $\mathfrak{m}$, such that

$$\nu(u_j) = \beta_j,\ 1 \leq j \leq n \tag{6}$$

and

$$\nu(x) = \max\{\gamma \in \Phi_x\},\ x \in A \setminus \{0\}. \tag{7}$$

This valuation is called the **monomial valuation** of A, associated to $\mathbf{u}$ and the n-tuple $(\beta_1, \dots, \beta_n)$. A valuation ν, with values in a group Γ, centered in $\mathfrak{m}$, is said to be **monomial with respect to u** if there exist $\beta_1, \dots, \beta_n \in \Gamma_+$ such that (7) holds for all $x \in A \setminus \{0\}$.

For further results on valuations, see also [43] or [45].

The following result is an immediate consequence of definitions:

Proposition 1.1.2 *Let G_ν be the graded algebra associated to a valuation $\nu : K \to \Gamma$, as above. Consider a sum of the form $y = \sum\limits_{i=1}^{s} y_i$, with $y_i \in K$. Let $\beta = \min\limits_{1 \leq i \leq s} \nu(y_i)$ and*

$$S = \{\, i \in \{1, \dots, n\} \mid \nu(y_i) = \beta\}\,.$$

The following two conditions are equivalent:
(1) $\nu(y) = \beta$
(2) $\sum\limits_{i \in S} in_\nu y_i \neq 0$.

1.2 Approximate roots up to $\nu(J)$ for an $\mathfrak{m}$-primary ideal J

Let A be a regular local ring of dimension n, $\mathfrak{m}$ its maximal ideal, $k = \frac{A}{\mathfrak{m}}$, $\mathbf{u} = (u_1, \dots, u_n)$ a regular system of parameters and

$$\nu : A \setminus \{0\} \to \Gamma$$

a valuation, centered in $\mathfrak{m}$ (this means $\nu(\mathfrak{m}) > 0$).

Let $\mathbf{1} = \nu(\mathfrak{m}) = \min\{\gamma \in \Phi \mid \gamma > 0\}$ and $\Phi_1 = \{\gamma \in \Phi \mid \exists a \in \mathbb{N}; \gamma < a \cdot \mathbf{1}\}$. For the sake of simplicity, we will write a instead of $a \cdot \mathbf{1}$. We shall study the structure of ν-ideals P_γ where $\gamma \in \Phi$.

If ν were monomial with respect to $\mathbf{u}$ then $\mathrm{in}_\nu u_1, \dots, \mathrm{in}_\nu u_n$ would generate $\mathrm{gr}_\nu A$ as a k-algebra. We are interested in analyzing valuations which are not necessarily monomial. We fix an $\mathfrak{m}$-primary valuation ideal J. The purpose of sections 1.2 and 1.3 is to construct a system of approximate roots up to $\nu(J)$, that is, a finite sequence of elements $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ of A such that for every ν-ideal I in A containing J we have

$$I = \left\{ \prod_j Q_j^{\gamma_j} \;\middle|\; \sum_j \gamma_j \nu(Q_j) \geq \nu(I) \right\} A \tag{8}$$

(in particular, the images $\mathrm{in}_\nu Q_i$ of the Q_i in $\mathrm{gr}_\nu A$ generate $\mathrm{gr}_\nu A$ as a k-algebra up to degree $\nu(J)$). In this construction, each Q_{i+1} will be described by an explicit formula (given later in this section) in terms of $Q_1, ..., Q_i$.

The earliest precursor of approximate roots appears in a series of papers by Saunders MacLane and O.F.G. Schilling [23], [24] and [25]. In dimension 2, they were defined globally in $k[x, y]$ by S. Abhyankar and T. T. Moh ([1], [2]) and locally by M. Lejeune-Jalabert [20]. See also the papers [18] and [19] by T. C. Kuo, [12] by R. Goldin and B. Teissier and [36] by M. Spivakovsky, [11] by F.J. Herrera Govantes, M.A. Olalla Acosta, M. Spivakovsky, [39]-[42] by Michel Vaquié. We also refer the reader to the paper [38] by B. Teissier for a different approach to the theory of approximate roots in higher dimensions.

Let $k = \dfrac{A}{\mathfrak{m}} = \dfrac{A}{\mathfrak{m}_\nu \cap A}$ be the residue field of A. Fix an isomorphism $\dfrac{A}{J} \cong \dfrac{k[u_1, \dots, u_n]}{J_0}$, where J_0 is an ideal of $k[u_1, \dots, u_n]$. In this way, we will view k as a subring of A/J.

We fix, once and for all, a section $k \to A$ of the natural map $A \to k$ which composed with the natural map $A \to \dfrac{A}{J}$ maps k isomorphically onto its image in $\dfrac{A}{J}$. The image of k in A will be denoted by $\mathbf{k}$.

According to Definition 0.1.17, we are looking for a finite set of elements $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$, $Q_i \in A$ satisfying (8).

Remark 1.2.1 *This means, in particular, that the initial forms $in_\nu(Q_1), in_\nu(Q_2), \dots$ generate $gr_\nu(A)$, up to degree $\nu(J)$. In other words, we want to build $\mathbf{Q}$ such that, for $f \in A$, we have $in_\nu(f) \in k[in_\nu \mathbf{Q}]$ provided $\nu(f) \leq \nu(J)$.*

Since J is an $\mathfrak{m}$-primary ideal, there are only finitely many elements of Φ less than or equal to $\nu(J)$. We proceed by induction on the finite set $\{\gamma \in \Phi \mid \gamma \leq \nu(J)\}$.

Definition 1.2.2 *Let E be an ordered set of elements of A. A generalized monomial $\mathbf{Q}^\alpha$ in E is a formal expression*

$$\mathbf{Q}^\alpha = \prod_{Q \in E} Q^{\alpha_Q}$$

where $\alpha_Q \in \mathbb{N}$ and $\alpha_Q = 0$ for all Q outside of a finite subset of E.

We view the set $\mathbb{N}^E$ as being ordrered lexicographically and order the set of generalized monomials by the lexicographical order of the pairs $(\nu(\mathbf{Q}^\alpha), \alpha)$.

The semigroup Φ is well ordered. For a natural number λ, γ_λ will denote the λ-th element of Φ.

We start by choosing a coordinate system adapted to the situation.

Definition 1.2.3 *Take $j \in \{2, \ldots, n\}$. We say that u_j is (ν, J)-prepared if either $u_j \in J$ or there does not exist $f \in A$ such that*

$$\begin{aligned} in_\nu u_j &= in_\nu f \quad \textit{and} \qquad (9)\\ f \mod J &\in \frac{k[u_1, \ldots, u_{j-1}]}{k[u_1, \ldots, u_{j-1}] \cap J_0}. \qquad (10)\end{aligned}$$

The coordinate system $\mathbf{u} = \{u_1, \ldots, u_n\}$ is (ν, J)-prepared if u_j is (ν, J)-prepared for all $j \in \{2, \ldots, n\}$.

Proposition 1.2.4 *There exists a (ν, J)-prepared coordinate system.*

Proof: We construct a (ν, J)-prepared coordinate system recursively in j. Assume that $u_2, \ldots, u_{j-1}$ are already (ν, J)-prepared, but u_j is not. Take $f \in A$ satisfying (9) and (10).

Let $\tilde{u}_j = u_j - f$; then $\nu(\tilde{u}_j) > \nu(u_j)$.

Since there are only finitely many elements of Φ less than $\nu(J)$, after finitely many repetitions of the above procedure, we may assume that u_j is (ν, J)-prepared. This completes the proof by induction on j. $\square$

We construct, recursively in λ, two finite ordered sets $\Lambda(\gamma_\lambda)$ and $\Theta(\gamma_\lambda)$ with

$$\Lambda(\gamma_\lambda) \subset \bigcup_{\lambda' < \lambda} \Theta(\gamma_{\lambda'}),$$

and a total ordering of the set $\Lambda(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$, compatible with the orders on $\Lambda(\gamma_\lambda)$ and $\Theta(\gamma_{\lambda-1})$. We do not impose a total order on the union $\bigcup_{\lambda'<\lambda} \Theta(\gamma_{\lambda'})$. At each step we define additional finite ordered sets

$$\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda), \qquad (11)$$

where the inclusions in (11) are inclusions of ordered sets. Both collections of sets $\Lambda(\gamma_\lambda)$ and $\mathcal{V}(\gamma_\lambda)$ will be increasing with λ. A typical element of each of those sets will have the form $(Q, \mathrm{Ex}(Q))$ where $Q \in A$ and $\mathrm{Ex}(Q)$ is a sum of monomials in $\Lambda(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$, written in the increasing order according to the on monomials, defined above.

Given an element $(Q, \mathrm{Ex}(Q)) \in \Lambda(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$, Q is called an *approximate root* and $\mathrm{Ex}(Q)$ is called the *expression* of Q. In what follows, we adopt the convention

$$\Theta(\gamma_\lambda) = \mathcal{V}(\gamma_\lambda) = \Psi(\gamma_\lambda) = \Lambda(\gamma_\lambda) = \emptyset$$

whenever $\lambda < 0$.

For a natural number ℓ, $\gamma_\ell \leq \nu(J)$, and for $(Q, \mathrm{Ex}(Q)) \in \Lambda(\gamma_\ell) \cup \Theta(\gamma_\ell)$, let In Q denote the smallest monomial of $\mathrm{Ex}(Q)$. Let

$$\mathrm{In}(\ell) = \left\{ \alpha \in \mathbb{N}^{\mathcal{V}(\gamma_\ell)} \;\middle|\; \exists (Q, \mathrm{Ex}(Q)) \in \Lambda(\gamma_\ell) \text{ such that } \mathbf{Q}^\alpha = \mathrm{In}\ Q \right\}.$$

Theorem 1.2.5 *For a natural number λ, $\gamma_\lambda \leq \nu(J)$, there exist finite ordered sets*

$$\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda)$$

and $\Theta(\gamma_\lambda)$ (and a total ordering of $\Lambda(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$) consisting of elements $(Q, Ex(Q))$, with $Q \in A$ and $Ex(Q)$ a sum of monomials in $\mathcal{V}(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$, increasing with respect to the given order on monomials, and having the following properties:

$$\begin{aligned} \nu(Q) < \gamma_\lambda &\textit{ whenever } (Q, Ex(Q)) \in \Lambda(\gamma_\lambda) \qquad (12)\\ \nu(Q) \geq \gamma_\lambda &\textit{ whenever } (Q, Ex(Q)) \in \Theta(\gamma_\lambda). \qquad (13)\end{aligned}$$

Moreover, for any $(Q, Ex(Q)) \in \Lambda(\gamma_\lambda)$, any monomial $\mathbf{Q}^\alpha$ appearing in $Ex(Q)$ is a monomial in $\mathcal{V}(\gamma_{\lambda-1})$ provided $Q \not\in \{u_1, \dots, u_n\}$. For any $(Q, Ex(Q)) \in \Theta(\gamma_\lambda)$, any monomial $\mathbf{Q}^\alpha$ appearing in $Ex(Q)$ is a monomial in $(\mathcal{V}(\gamma_{\lambda+1}) \cap \Theta(\gamma_{\lambda-1})) \cup \mathcal{V}(\gamma_\lambda)$ provided $Q \not\in \{u_1, \dots, u_n\}$. An element

$$(Q, Ex(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$$

is completely determined by In Q.

Proof: We proceed by induction on λ.

First define $\Psi(\mathbf{1}) = \Lambda(\mathbf{1}) = \emptyset$ and $\Theta(\mathbf{1}) = \{(u_1, u_1), \dots, (u_n, u_n)\}$ where we assume

$$\nu(u_1) \leq \nu(u_2) \leq \cdots \leq \nu(u_n).$$

We define the total ordering on $\Theta(\mathbf{1})$ by $(u_1, u_1) < (u_2, u_2) < \cdots < (u_n, u_n)$.

Let $\lambda > 0$ be a natural number such that $\gamma_\lambda \leq \nu(J)$. Assume that for each $\ell < \lambda$ we have constructed sets $\mathcal{V}(\gamma_\ell) \subset \Psi(\gamma_\ell) \subset \Lambda(\gamma_\ell)$ and $\Theta(\gamma_\ell)$ having the properties required in the theorem.

Let

$$\Lambda(\gamma_\lambda) = \Lambda(\gamma_{\lambda-1}) \cup \{(Q, \mathrm{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid \nu(Q) < \gamma_\lambda\}. \tag{14}$$

Definition 1.2.6 *An element $(Q, Ex(Q)) \in \Lambda(\gamma_\lambda)$ is an **inessential predecessor** of an approximate root $(Q', Ex(Q')) \in \Lambda(\gamma_\lambda)$ if $Ex(Q') = Ex(Q) + \sum_\alpha c_\alpha \mathbf{Q}^\alpha$, where $c_\alpha \in k$ and the $\mathbf{Q}^\alpha$ are monomials in $\mathcal{V}(\gamma_\lambda)$.*

*An element $(Q, Ex(Q)) \in \Lambda(\gamma_\lambda)$ is said to be **essential at the level** γ_λ if Q is not an inessential predecessor of an element of $\Lambda(\gamma_\lambda)$.*

Let $\Psi(\gamma_\lambda)$ be the subset of $\Lambda(\gamma_\lambda)$ consisting of all the essential roots at the level γ_λ. Let $\mathcal{V}(\gamma_\lambda)$ be the subset of $\Psi(\gamma_\lambda)$ consisting of all $(Q, \mathrm{Ex}(Q))$ such that $\mathrm{in}_\nu(Q)$ does not belong to the k-vector space of $G = \mathrm{gr}_\nu(A)$ generated by the set $\{\mathrm{in}_\nu \mathbf{Q}^\gamma\}$ where $\mathbf{Q}^\gamma$ runs over the set of all the generalized monomials on roots preceding Q in the above ordering.
We extend the total ordering from $\Lambda(\gamma_{\lambda-1})$ to $\Lambda(\gamma_\lambda)$ by postulating that $\Lambda(\gamma_{\lambda-1})$ is the initial segment of $\Lambda(\gamma_\lambda)$. Moreover, we extend this order to $\Lambda(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$ by postulating that $\Lambda(\gamma_\lambda)$ is the initial segment of $\Lambda(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$.

For a natural number ℓ, let $E(\ell) = \mathrm{In}(\ell) + \mathbb{N}^{\mathcal{V}(\gamma_\ell)} \subset \mathbb{N}^{\mathcal{V}(\gamma_\ell)}$.

Now consider the ordered set $\{\mathbf{Q}^{\alpha_1}, \dots, \mathbf{Q}^{\alpha_s}\}$ of monomials

$$\mathbf{Q}^\alpha = \prod Q^{\alpha_Q},\ (Q, \mathrm{Ex}(Q)) \in \mathcal{V}(\gamma_\lambda) \cup \{\, (Q, \mathrm{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid\ \nu(Q) = \gamma_\lambda\} \tag{15}$$

of value γ_λ, such that the natural projection of α to $\mathbb{N}^{\mathcal{V}(\gamma_\lambda)}$ does not belong to $E(\lambda)$.

Let $i_1 = \max\left\{ i \in \{1, \dots, s\} \;\middle|\; \mathrm{in}_\nu(\mathbf{Q}^{\alpha_i}) \in \sum\limits_{j=i+1}^{s} k\ \mathrm{in}_\nu(\mathbf{Q}^{\alpha_j}) \right\}$ and consider the unique relation $\mathrm{in}_\nu(\mathbf{Q}^{\alpha_{i_1}}) - \sum\limits_{j=i_1+1}^{s} \overline{c_{1j}}\ \mathrm{in}_\nu(\mathbf{Q}^{\alpha_j}) = 0$. Let $P_1 = \mathbf{Q}^{\alpha_{i_1}} - \sum\limits_{j=i_1+1}^{s} c_{1j} \mathbf{Q}^{\alpha_j}$ where $c_{1j} \in \mathbf{k}$ is the image of $\overline{c_{1j}}$ under the chosen section $k \to A$.

Let $i_2 = \max\left\{ i \in \{1, \dots, i_1 - 1\} \;\middle|\; \mathrm{in}_\nu(\mathbf{Q}^{\alpha_i}) \in \sum\limits_{j=i+1}^{s} k\ \mathrm{in}_\nu(\mathbf{Q}^{\alpha_j}) \right\}$ and, as before, consider the unique $P_2 = \mathbf{Q}^{\alpha_{i_2}} - \sum\limits_{\substack{j=i_2+1 \\ j \neq i_1}}^{s} c_{2j} \mathbf{Q}^{\alpha_j}$ such that the vector $(\alpha_j)_{j=i_1+1,\dots,s}$, $\overline{c_{2j}} \neq 0$, is minimal in the lexicographical order. We continue in this way and define $P_3, \dots, P_t$.

Let

$$\Theta(\gamma_\lambda) = \{\, (Q, \mathrm{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid\ \nu(Q) \geq \gamma_\lambda\} \cup \{(P_1, \mathrm{Ex}(P_1)), \dots, (P_t, \mathrm{Ex}(P_t))\} \tag{16}$$

where

$$\mathrm{Ex}(P_j) = \mathrm{Ex}(Q) - \sum_k c_{jk}\mathbf{Q}^{\alpha_k} \tag{17}$$

if $\mathbf{Q}^{\alpha_{i_j}} = Q$ with $(Q, \mathrm{Ex}(Q)) \in \{(Q, \mathrm{Ex}(Q)) \in \Theta(\gamma_{\lambda-1}) \mid \nu(Q) = \gamma_\lambda\}$ and

$$\mathrm{Ex}(P_j) = \mathbf{Q}^{\alpha_{i_j}} - \sum_k c_{jk}\mathbf{Q}^{\alpha_k} \tag{18}$$

otherwise.

We define the order on $\Theta(\gamma_\lambda)$ by $\Theta(\gamma_{\lambda-1}) < \{(P_1, \mathrm{Ex}(P_1)), \dots, (P_t, \mathrm{Ex}(P_t))\}$ and $(P_1, \mathrm{Ex}(P_1)) < \dots < (P_t, \mathrm{Ex}(P_t))$.

Remark 1.2.7 *Note that, because the coordinate system is prepared, $u_1, \dots, u_n$ are always essential.*

Remark 1.2.8 *Suppose given two approximate roots Q_1 and Q_2 such that*

$$In(Q_1) = In(Q_2) = \mathbf{Q}^\alpha$$

and suppose that Q_1 appears before Q_2 in the process of construction of the approximate roots decribed above. Because of the uniqueness of the construction of the P_i's above, we have

$$\nu(Q_2) > \nu(Q_1).$$

Now, if $\nu(\mathbf{Q}^\alpha) = \gamma_\ell$, then $\alpha \in E(\ell)$, so the only way the monomial $\mathbf{Q}^\alpha$ can appear as an initial form of Q_2 is when $P_k = Q' + \sum c_j\mathbf{Q}^{\alpha_j}$ where $In(Q') = \mathbf{Q}^\alpha$ and then $\nu(Q') < \nu(Q_2)$. Then, either $\nu(Q') = \nu(Q_1)$ and so $Q' = Q_1$ because of the uniqueness in the construction process, or $\nu(Q') > \nu(Q_1)$, but we conclude by descending induction that $Q_2 = Q_1 + \sum c_j\mathbf{Q}^{\alpha_j}$ and $Ex(Q_2) = Ex(Q_1) + \sum c_j\mathbf{Q}^{\alpha_j}$.

So finally, the expression of an approximate root has the form

$$\mathrm{Ex}(Q) = \mathbf{Q}^{\alpha_{i_j}} + \sum_k a_k\mathbf{Q}^{\alpha_k} \tag{19}$$

the sum being written in the increasing order of the monomials.

Remark 1.2.9 *This construction is very similar to finding a basis of the space of relations by row reduction.*

Remark 1.2.10 *We just showed that there is a one to one correspondence $Q \leftrightarrow In(Q)$ between the approximate roots $Q \in \Psi(\gamma_\ell)$ and the set of monomials which are the first term of the expression $Ex(Q)$ of such an approximate root Q. Let us denote by $\mathbf{M}(\ell)$ the set of those monomials.*

The last part of the theorem holds by construction. □

1.3 Standard form up to $\nu(J)$

Consider the integer λ such that $\gamma_\lambda = \nu(J)$. Assume that the system of coordinates u of A is (ν, J)-prepared.

Definition 1.3.1 *A monomial in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ is called **standard with respect to** λ if all the approximate roots appearing in it belong to $\mathcal{V}(\gamma_\lambda)$ and it is not divisible by any $In(Q)$ where Q is an approximate root in $(\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \dots, (u_n, u_n)\}$.*

Definition 1.3.2 *Let $f \in A$ and let ℓ be a positive integer, $\ell \leq \lambda$. An expression of the form*

$$f = \sum c_\alpha \mathbf{Q}^\alpha,$$

where the $\mathbf{Q}^\alpha$ are monomials in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$, written in the increasing order, is ***a standard form of level*** *γ_ℓ with respect to λ if for all $\gamma' < \gamma_\ell$ and for all α such that $\nu(\mathbf{Q}^\alpha) = \gamma'$ and $c_\alpha \neq 0$, $\mathbf{Q}^\alpha$ is a standard monomial with respect to λ.*

We now construct, by induction on ℓ, a standard form of f of level γ_ℓ. We will write this standard form as

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

where, for all α, $\mathbf{Q}^\alpha$ is a generalized monomial in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$, $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$ and f_ℓ is a sum of standard monomials in $\mathcal{V}(\gamma_\lambda)$ of value strictly less than γ_ℓ.

To start the induction, let $f_0 = 0$. The *standard form of f of level 0 with respect to λ* will be its expansion $f = f_0 + \sum c_\alpha \mathbf{u}^\alpha$ as a formal power series in the u_i, with the monomials written in the increasing order according to the monomial order defined above,.

Let ℓ be a natural number, $\ell < \lambda$. Let us define $f_{\ell+1}$ and the standard form of f of level $\gamma_{\ell+1}$ as follows. Assume we already have an expression $f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$ with $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$, for all α, and the value of any monomial of f_ℓ is strictly less than γ_ℓ.

Take the homogeneous part of $\sum c_\alpha \mathbf{Q}^\alpha$ of value γ_ℓ, with the monomials arranged in the increasing order, and consider the first monomial $\mathbf{Q}^\alpha$ which is not standard. Since $\mathbf{Q}^\alpha$ is not standard, one of the following two conditions holds:

1. There exists an approximate root $Q \in (\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \dots, (u_n, u_n)\}$ such that $In(Q)$ divides $\mathbf{Q}^\alpha$. Write $Q = In(Q) + \sum c_\beta \mathbf{Q}^\beta$ and replace $In(Q)$ by $Q - \sum c_\beta \mathbf{Q}^\beta$ in $\mathbf{Q}^\alpha$.
2. There exists $Q \in \Psi(\gamma_\lambda) \setminus \mathcal{V}(\gamma_\lambda)$ which divides $\mathbf{Q}^\alpha$. Since $Q \notin \mathcal{V}(\gamma_\lambda)$, there exists

   $$Q' \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$$

   of the form $Q' = Q + \sum\limits_\delta d_\delta \mathbf{Q}^\delta$ where $\mathbf{Q}^\delta$ are monomials in $\mathcal{V}(\gamma_\lambda)$ of value greater than or equal to γ_ℓ. Replace Q by $Q' - \sum\limits_\delta d_\delta \mathbf{Q}^\delta$.

In both cases, those changes introduce new monomials, but either they are of value strictly greater than γ_ℓ or they are of value exactly γ_ℓ but greater than $\mathbf{Q}^\alpha$ in the monomial ordering. We repeat this procedure as many times as we can. After a finite number of steps, no more changes are available at level $\gamma_{\ell+1}$. Then, let $f_{\ell+1} = f_\ell + \sum d_\beta \mathbf{Q}^\beta$ with $\nu(\mathbf{Q}^\beta) = \gamma_\ell$, so that $f = f_{\ell+1} + \sum c_\alpha \mathbf{Q}^\alpha$ where $\nu(\mathbf{Q}^\alpha) > \gamma_\ell$.

The expression thus constructed satisfies the definition of standard form of level $\gamma_{\ell+1}$ because all the non-standard monomials $\mathbf{Q}^\alpha$ of value less than or equal to γ_ℓ have been eliminated.

Proposition 1.3.3 *Let*

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

be a standard form of f of level γ_ℓ and $\gamma < \gamma_\ell$ an element of Φ. Then $\sum\limits_{\nu(\mathbf{Q}^\beta)=\gamma} c_\beta \mathbf{Q}^\beta \notin P_{\gamma+}$.

Proof : We give a proof by contradiction. Suppose there exists a relation of the form

$$\sum_{\nu(\mathbf{Q}^\beta)=\gamma} c_\beta \mathbf{Q}^\beta \in P_{\gamma+}. \tag{20}$$

Let $\mathbf{Q}^\alpha$ be the smallest monomial on the left hand side of (20). By construction of approximate roots, there exists a finite collection $Q_1, \dots, Q_s \in \Lambda(\gamma+) \cup \theta(\gamma+)$ and generalized monomials $\mathbf{Q}^{\omega_1}, \dots, \mathbf{Q}^{\omega_s}$ such that $\sum\limits_{i=1}^{s} Q_i \mathbf{Q}^{\omega_i} = \sum\limits_{\nu(\mathbf{Q}^\beta)=\gamma} c_\beta \mathbf{Q}^\beta$.

There exists $i \in \{1, \ldots, s\}$ such that one of the two conditions holds : either

$$\mathbf{Q}^{\alpha} = \mathbf{Q}^{\omega_i} \cdot In(Q_i)$$

or

$$Q_i = Q'_i + \sum_{\epsilon} b_\epsilon \mathbf{Q}^{\epsilon},\ Q'_i \in \Lambda(\gamma+) \setminus \Psi(\gamma+).$$

In either case, the monomial $\mathbf{Q}^{\alpha}$ is not standard, which gives the desired contradiction. □

For each ℓ, the part f_ℓ of a standard form of f of level γ_ℓ is uniquely determined. This is a straightforward consequence of the Proposition.

As a consequence of Proposition 1.3.3, note that if $\gamma_\ell > \nu(f)$ then $\nu(f)$ equals the smallest value of a monomial appearing in the standard form of f of level γ_ℓ.

Theorem 1.3.4 *(1) Take $\gamma \in \Phi$, $\gamma < \gamma_\lambda$. Then $\dfrac{P_\gamma}{P_{\gamma+}}$ is generated as a k-vector space by $\{in_\nu \mathbf{Q}^{\beta}\}$ where $\mathbf{Q}^{\beta}$ runs over the set of all the standard monomials with respect to λ, satisfying $\nu(\mathbf{Q}^{\beta}) = \gamma$.*

(2) The part of the graded k-algebra $gr_\nu(A)$ of degree strictly less than γ_λ is generated by the initial forms of the approximate roots of $\mathcal{V}(\gamma_\lambda)$.

Proof : Take an element $\gamma \in \Phi$, $\gamma < \gamma_\lambda$. Let $h \in P_\gamma/P_{\gamma^+}$ be a homogeneous element of degree γ of $\mathrm{gr}_\nu(A)$ and let $f \in P_\gamma$ be such that $\mathrm{in}_\nu(f) = h$. Let $\sum c_\beta \mathbf{Q}^{\beta}$ denote the homogeneous part of least value of a standard form of f of level γ_λ. Then the initial form of f is $\sum \mathrm{in}_\nu(c_\beta \mathbf{Q}^{\beta})$. □

The Alvis–Johnston–Madden example. Let α be the point of $\mathrm{Sper}(R[x,y,z])$ given by the curvette $x(t) = t^6,\ y(t) = t^{10} + ut^{11}, z(t) = t^{14} + t^{15}$ where u is some fixed element of R with $u > 2$. Let J be a ν_α-ideal of value greater than or equal to 37.
The calculation of the first few approximate roots gives

$$\begin{aligned}
Q_1 &= x, && (21)\\
Q_2 &= y, && (22)\\
Q_3 &= z, && (23)\\
Q_4 &= y^2 - xz = (2u-1)t^{21} + u^2t^{22},\ \nu(Q_4) = 21 && (24)\\
Q_5 &= yz - x^4 = (u+1)t^{25} + ut^{26},\ \nu(Q_5) = 25 && (25)\\
Q_6 &= z^2 - x^3y = (2-u)t^{29} + t^{30},\ \nu(Q_6) = 29 && (26)\\
Q_7^{(31)} &= yQ_4 - \alpha(u)xQ_5,\ \alpha(u) = (2u-1)/(u+1),\ \nu\left(Q_7^{(31)}\right) = 32 && (27)\\
Q_7^{(32)} &= yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z,\ \nu\left(Q_7^{(32)}\right) = 33 && (28)
\end{aligned}$$

$$\begin{aligned}
Q_7^{(33)} &= yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z - \gamma(u)x^2Q_4 && (29)\\
Q_7^{(34)} &= yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z - \gamma(u)x^2Q_4 - \delta(u)x^4y && (30)\\
Q_7^{(35)} &= yQ_4 - \alpha(u)xQ_5 - \beta(u)x^3z - \gamma(u)x^2Q_4 - \delta(u)x^4y - \epsilon(u)xQ_6 && (31)\\
Q_8^{(35)} &= zQ_4 + \zeta(u)xQ_6 && (32)\\
Q_9^{(35)} &= yQ_5 + \eta(u)xQ_6, && (33)
\end{aligned}$$

where $\beta(u)$, $\gamma(u)$, $\delta(u)$, $\epsilon(u)$, $\zeta(u)$, $\eta(u)$ are functions of u which can be calculated explicitly.

The elements listed above belong to $\Lambda(37)$; we chose to index them as $Q_i^{(j)}$. In this notation, the approximate root $Q_i^{(j)}$ is an inessential predecessor of $Q_i^{(j+1)}$ whenever $Q_i^{(j+1)}$ is defined.

We also note the relation $xQ_6 - yQ_5 + zQ_4 = 0$, which is the simplest example of a syzygy, an important phenomenon, responsible for much of the difficulty of the Pierce–Birkhoff conjecture.

In the same vein, we can describe the standard form of different levels of an element of A, say for instance,

$$f = x^3 + y^3 + z^3 \tag{34}$$

(which is a standard form of level 0). For $\gamma \leq 30$, the standard form of f of level γ is given by (34). Then, as $y^2 \in E(8)$ (this is so because 21 is the eighth positive element of the value semigroup Φ), we replace y^3 by $y(Q_4 + xz)$ to obtain

$$f = x^3 + yQ_4 + xyz + z^3. \tag{35}$$

Since $yz \in E(11)$ (note that 25 is the eleventh positive element of the value semigroup Φ), we replace xyz in (35) by $xQ_5 + x^5$, to obtain the standard form of level 31:

$$f = x^3 + x^5 + yQ_4 + xQ_5 + z^3 \tag{36}$$

(the monomials being written in the order of increasing values 18, 30, 31, 31, 42). Next, we replace yQ_4 by $\alpha(u)xQ_5$ in (36), so the standard form of levels 32, 33, 34 and 35 is given by

$$f = x^3 + x^5 + (1 + \alpha(u))xQ_5 + \beta(u)x^3 z + \gamma(u)x^2 Q_4 + \delta(u)x^4 y + Q_7^{(34)} + z^3,$$

and so on ...

Let ℓ be an integer such that $\gamma_\ell \leq \gamma_\lambda$. Let $X = X_{\mathcal{V}(\gamma_\ell)}$ be a set of independent variables, indexed by $\mathcal{V}(\gamma_\ell)$, and consider the graded k-algebra $k\left[X_{\mathcal{V}(\gamma_\ell)}\right]$, where we define

$$\deg\ X_j = \nu(Q_j).$$

Let P denote the homogeneous monomial ideal of $k\left[X_{\mathcal{V}(\gamma_\ell)}\right]$ generated by all the monomials in $X_{\mathcal{V}(\gamma_\ell)}$ of degree greater than or equal to γ_ℓ. We have a natural map

$$\begin{array}{rcl} \phi_\ell : \frac{k\left[X_{\mathcal{V}(\gamma_\ell)}\right]}{P} & \to & \frac{\mathrm{gr}_\nu A}{P_{\gamma_\ell}} \\ X_j & \mapsto & \mathrm{in}_\nu Q_j. \end{array}$$

Now, for $\ell = 0$, let $I_0 = (0)$. For $\ell > 0$, let I_ℓ denote the ideal of $\frac{k\left[X_{\mathcal{V}(\gamma_\ell)}\right]}{P}$ generated by all the homogeneous polynomials of the form

$$X^{\alpha_0} + \lambda_1 X^{\alpha_1} + \lambda_2 X^{\alpha_2} + \dots + \lambda_{b_0} X^{\alpha_{b_0}} \tag{37}$$

where $\mathbf{Q}^{\alpha_0} + \lambda_1 \mathbf{Q}^{\alpha_1} + \lambda_2 \mathbf{Q}^{\alpha_2} + \dots + \lambda_{b_0} \mathbf{Q}^{\alpha_{b_0}}$ is the homogeneous part of least degree of $Ex(Q)$ for an approximate root $Q \in \mathcal{V}(\gamma_\ell) \cup \Theta(\gamma_\ell)$.

Corollary 1.3.5 *We have* $\ker\ \phi_\ell = I_\ell$.

Proof : The inclusion $I_\ell \subset \ker\ \phi_\ell$ is immediate. To prove the opposite inclusion, we argue by contradiction. Take a homogeneous element

$$h = a_{\lambda_1} X^{\lambda_1} + a_{\lambda_2} X^{\lambda_2} + \dots + a_{\lambda_s} X^{\lambda_s} \in \ker(\phi_\ell) \setminus I_\ell \tag{38}$$

of degree b, $b < \gamma_\ell$, such that λ_1 is lexicographically smallest among all the elements $h \in \ker(\phi_\ell) \setminus I_\ell$ of degree b.

The inclusion (38) implies that

$$a_{\lambda_1} \mathrm{in}_\nu \mathbf{Q}^{\lambda_1} + a_{\lambda_2} \mathrm{in}_\nu \mathbf{Q}^{\lambda_2} + \dots + a_{\lambda_s} \mathrm{in}_\nu \mathbf{Q}^{\lambda_s} = 0. \tag{39}$$

in P_b/P_{b+}.

By definition of I_ℓ, there exists an element $g \in I_\ell$ of the form $X^\epsilon + \sum_p c_p X^{\epsilon_p}$ and a monomial X^δ with $\epsilon_p > \epsilon$ for all p and $\lambda_1 = \epsilon + \delta$. Then, as $g \in I_\ell \subset \ker(\phi_\ell)$, we have $h - a_{\lambda_1} X^\delta g \in \ker(\phi_\ell)$ and the greatest monomial of $h - a_{\lambda_1} X^\delta g$ is strictly bigger than X^{λ_1}. This contradicts the choice of h. $\square$

Corollary 1.3.6 *Take an element $\gamma \in \Phi$, $\gamma < \gamma_\lambda$. The valuation ideal P_γ is generated by all the generalized monomials of value greater than or equal to γ in $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda)\}$. The ideal P_{γ_λ} is generated by all the generalized monomials of value greater than or equal to γ_λ in $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)\}$.*

Proof: Let $f \in P_\gamma$ (resp. $f \in P_{\gamma_\lambda}$). By the very definition of the standard form of level ℓ such that $\gamma_\ell = \gamma$, f can be written as an A-linear combination of generalized monomials of value greater than or equal to γ in $\{Q \mid (Q, Ex(Q)) \in \Psi(\gamma_\lambda)\}$ (resp. $\in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$). Thus P_γ (resp. P_{γ_λ}) is generated by the generalized monomials of value at least γ, as desired. $\square$

1.4 Approximate roots in a complete regular local ring

We now generalize the notion of approximate root to a **complete** regular local ring A of dimension n, with maximal ideal $\mathfrak{m}$, and residue field $k = \dfrac{A}{\mathfrak{m}}$. Let $\mathbf{u} = (u_1, \dots, u_n)$ be a regular system of parameters and

$$\nu : A \setminus \{0\} \to \Gamma$$

a valuation, centered in $\mathfrak{m}$. Denote by $\nu_{\mathfrak{m}}$ the $\mathfrak{m}$-adic valuation.

We keep the same notation as in §2.

The purpose of this section is to construct, for a general ν, a system of approximate roots of ν, that is, a well-ordered collection of elements $\mathbf{Q} = \{Q_i\}_{i \in \Lambda}$ of A such that for every ν-ideal I in A, we have

$$I = \left\{ \prod_j Q_j^{\gamma_j} \;\middle|\; \sum_j \gamma_j \nu(Q_j) \geq \nu(I) \right\} A \tag{40}$$

(in particular, the images $\mathrm{in}_\nu Q_i$ of the Q_i in $\mathrm{gr}_\nu A$ generate $\mathrm{gr}_\nu A$ as a k-algebra). Each Q_{i+1} will be described by an explicit formula (given later in this section) in terms of the Q_j, $j < i$.

In this general setting, we have to proceed by transfinite induction on the well-ordered semigroup Φ. Since we are not assuming that $rk\ \Gamma = 1$ or that Φ is Archimedean, we have to work with ordinals other than the natural numbers.

Remark on the use of transfinite induction. Since the ring A is noetherian, the group Γ of values of ν has finite rank. Therefore all the ordinals ℓ we will encounter in this paper will be of type $\ell \leq \omega^n$ (cf. [43] and [8]). Thus we will be using a very special form of transfinite induction, which amounts to usual induction, applied finitely many times. We will, however, stick to the language of transfinite induction to simplify the exposition.

Recall the definition of generalized monomial with respect to a totally ordered set $E \subset A$ (Definition 1.2.2). Assume in addition that E is well-ordered. We well-order the set $\mathbb{N}^E$ by the lexicographical ordering and the set of generalized monomials by the lexicographical ordering on the set of triples $(\nu(\mathbf{Q}^\alpha), \nu_{\mathfrak{m}}(\mathbf{Q}^\alpha), \alpha)$.

The semigroup Φ is well ordered. By abuse of notation, we will sometimes write Φ for the ordinal given by the order type of Φ. Let $\lambda < \Phi$ be an ordinal and γ_λ the element of Φ corresponding to λ.

We start by choosing a coordinate system adapted to the situation. Fix an isomorphism

$$A \cong k[[u_1, \dots, u_n]]. \tag{41}$$

Definition 1.4.1 *Take $j \in \{2, \dots, n\}$. We say that u_j is ν-prepared if there does not exist $f \in A$ such that $in_\nu u_j = in_\nu f$ and $f \in k[[u_1, \dots, u_{j-1}]]$. The coordinate system $\mathbf{u} = \{u_1, \dots, u_n\}$ is ν-prepared if u_j is ν-prepared for all $j \in \{2, \dots, n\}$.*

Proposition 1.4.2 *There exists a ν-prepared coordinate system.*

Proof: We construct a ν-prepared coordinate system recursively in j. Assume that $u_2, \ldots, u_{j-1}$ are ν-prepared, but u_j is not.

We will construct the prepared coordinate $\tilde{u}_j$ recursively by transfinite induction on Φ. More precisely, we will construct a well ordered set $\{u_{ji}\}$ of successive approximation to $\tilde{u}_j$ in the $\mathfrak{m}$-adic topology. We will show that this set satisfies the hypothesis of Zorn's lemma and let $\tilde{u}_j$ be its maximal element.

The details go as follows. Let $u_{j0} = u_j$. Suppose that u_{ji} is constructed and that it is not prepared. Let f_{ji} be the element f of $k[[u_1, \ldots, u_{j-1}]]$ appearing in the definition of "not prepared". Put $u_{j,i+1} = u_{ji} - f_{ji}$. Then $\nu(u_{ji}) = \nu(f_{ji}) < \nu(u_{j,i+1})$. Next, suppose given a sequence $u_{ji}, u_{j,i+1}, \ldots$ of elements of $k[[u_1, \ldots, u_j]]$ such that $(u_1, \ldots, u_{j-1}, u_{jq})$ is a regular system of parameters of $k[[u_1, \ldots, u_j]]$ for each q and

$$\nu(u_{ji}) < \nu(u_{j,i+1}) < \nu(u_{j,i+2}) < \cdots .$$

Let $\beta_q = \nu(u_{jq})$. Since the ring A is noetherian, the semi-group Φ is well-ordered. Let $\bar{\beta} = \min\{\beta \in \Phi \mid \beta > \beta_q, \forall q \in \mathbb{N}\}$. By Chevalley's lemma, applied to the nested sequence of ideals $\dfrac{P_{\beta_q} \cap k[[u_1, \ldots, u_{j-1}]]}{P_{\bar{\beta}} \cap k[[u_1, \ldots, u_{j-1}]]}$ in the complete local ring $\dfrac{k[[u_1, \ldots, u_{j-1}]]}{P_{\bar{\beta}} \cap k[[u_1, \ldots, u_{j-1}]]}$, we see that $\lim\limits_{q \to \infty} (f_{jq} \mod P_{\bar{\beta}}) = 0$ in the $(u_1, \ldots, u_{j-1})$-adic topology.

Hence, modifying each f_{jq} by an element of $P_{\bar{\beta}}$ if necessary, we may assume that

$$\lim_{q \to \infty} f_{jq} = 0.$$

We define $u_{j,i+\omega}$ to be the formal power series $u_{ji} - f_{ji} - f_{j,i+1} - \cdots$. By construction,

$$\nu(u_{j,i+\omega}) \geq \bar{\beta}.$$

To complete our construction , we need to consider countable well ordered sets $\{u_{jt}\}$ of order type greater than ω. This presents no problem: by countability, we can always choose a cofinal subsequence in each such set. Then the above construction of $u_{j,i+\omega}$ applies verbatim. $\Box$

We construct, inductively in λ, two well-ordered sets $\Lambda(\gamma_\lambda)$ and $\Theta(\gamma_\lambda)$ and, in the case λ is not a limit ordinal, a well ordering of the set $\Lambda(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$, compatible with the orders on $\Lambda(\gamma_\lambda)$ and $\Theta(\gamma_{\lambda-1})$. At each step we define two additional well-ordered sets $\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda)$ where the inclusions are inclusions of ordered sets. Both collections of sets $\Lambda(\gamma_\lambda)$ and $\mathcal{V}(\gamma_\lambda)$ will be increasing with λ.

A typical element of each of those sets will have the form $(Q, \mathrm{Ex}(Q))$ where $Q \in A$ and $\mathrm{Ex}(Q)$ is an increasing sum of monomials in $\mathcal{V}(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$ if λ is not a limit ordinal, resp. monomials in $\mathcal{V}(\gamma_\lambda)$ if λ is a limit ordinal. The sum in $\mathrm{Ex}(Q)$ may be finite or infinite, but it is always convergent in the $\mathfrak{m}$-adic topology. Given an element $(Q, \mathrm{Ex}(Q)) \in \Lambda(\gamma_\Phi) \cup \Theta(\gamma_\Phi)$, Q is called an *approximate root* and $\mathrm{Ex}(Q)$ is called the *expression* of Q.

For an ordinal $\ell < \Phi$ and for $(Q, \mathrm{Ex}(Q)) \in \Lambda(\gamma_\ell) \cup \Theta(\gamma_\ell)$, let In Q denote the smallest monomial of $\mathrm{Ex}(Q)$. Let $\mathrm{In}(\ell) = \left\{ \alpha \in \mathbb{N}^{\mathcal{V}(\gamma_\ell)} \;\middle|\; \exists (Q, \mathrm{Ex}(Q)) \in \Lambda(\gamma_\ell) \text{ such that } \mathbf{Q}^\alpha = \mathrm{In}\ Q \right\}$.

Theorem 1.4.3 *For $\lambda < \Phi$, there exist well ordered sets $\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda) \subset \Lambda(\gamma_\lambda)$ and $\Theta(\gamma_\lambda)$, and a well ordering of $\Lambda(\gamma_\lambda) \cup \Theta(\gamma_{\lambda-1})$ when λ is not a limit ordinal, having the following properties. Let*

$$\Psi(<\gamma_\lambda) = \Psi(\gamma_{\lambda-1}) \quad \textit{if } \lambda \textit{ is not a limit ordinal and} \qquad (42)$$
$$\Psi(<\gamma_\lambda) = \Psi(\gamma_\lambda) \quad \textit{otherwise} \qquad (43)$$

and similarly for $\mathcal{V}(<\gamma_\lambda)$. Then each set $\mathcal{V}(\gamma_\lambda), \Psi(\gamma_\lambda),\ \Lambda(\gamma_\lambda),\ \Theta(\gamma_\lambda)$ consists of elements of the form $(Q, Ex(Q))$, with $Q \in A$ and $Ex(Q)$ is an increasing (with respect to the monomial *order defined above) sum of monomials in $\mathcal{V}(<\gamma_\lambda)\cup\Theta(\gamma_{\lambda-1})$ when λ is not a limit ordinal, resp. $\mathcal{V}(\gamma_\lambda)$ when λ is a limit ordinal, of value $<\nu(Q)$, provided $Q\not\in\{u_1,\dots,u_n\}$, such that*

$$\nu(Q)<\gamma_\lambda \text{ whenever } (Q,Ex(Q))\in\Lambda(\gamma_\lambda) \tag{44}$$
$$\nu(Q)\geq\gamma_\lambda \text{ whenever } (Q,Ex(Q))\in\Theta(\gamma_\lambda) \tag{45}$$

and the sets

$$\{(Q,Ex(Q))\in\Theta(\gamma_\lambda)\cup\Lambda(\gamma_\lambda)\mid\nu(Q)=\gamma\},\ \gamma\in\Phi \tag{46}$$

and

$$\{\,(Q,Ex(Q))\in\Psi(\gamma_\lambda)\cup\Theta(\gamma_\lambda)\mid Q\not\in\mathfrak{m}^s\},\ s\in\mathbb{N} \tag{47}$$

are finite. An element $(Q,Ex(Q))\in\Psi(\gamma_\lambda)\cup\Theta(\gamma_\lambda)$ is completely determined by $In\ Q$; moreover $\nu_{\mathfrak{m}}(In\ Q)=\nu_{\mathfrak{m}}(Q)$.

In what follows, $\Lambda(<\gamma_\lambda)$ will stand for $\bigcup\limits_{\ell<\lambda}\Lambda(\gamma_\ell)$.

Proof : We proceed by transfinite induction.

First define $\Psi(\mathbf{1})=\Lambda(\mathbf{1})=\emptyset$ and $\Theta(\mathbf{1})=\{(u_1,u_1),\dots,(u_n,u_n)\}$ where we assume

$$\nu(u_1)\leq\nu(u_2)\leq\cdots\leq\nu(u_n).$$

We define the well ordering on $\Theta(\mathbf{1})$ by $(u_1,u_1)<(u_2,u_2)<\cdots<(u_n,u_n)$.

Let $\lambda<\Phi$ be an ordinal. Assume that for each $\ell<\lambda$ we have constructed sets $\Psi(\gamma_\ell)\subset\Lambda(\gamma_\ell)$ and $\Theta(\gamma_\ell)$ and a well ordering of $\Lambda(\gamma_\ell)\cup\Theta(\gamma_{\ell-1})$, having the properties required in the theorem.

Let

$$\Lambda(\gamma_\lambda) = \Lambda(<\gamma_\lambda) \text{ if } \lambda \text{ is a limit ordinal} \tag{48}$$
$$\Lambda(\gamma_\lambda) = \Lambda(\gamma_{\lambda-1})\cup\{(Q,\mathrm{Ex}(Q))\in\Theta(\gamma_{\lambda-1})\mid\nu(Q)<\gamma_\lambda\} \text{ otherwise.} \tag{49}$$

Definition 1.4.4 *An element $(Q,Ex(Q))\in\Lambda(\gamma_\lambda)$ is an* ***inessential predecessor*** *of a root $(Q',Ex(Q'))\in\Lambda(\gamma_\lambda)$ if $Ex(Q')=Ex(Q)+\sum_\alpha c_\alpha\mathbf{Q}^\alpha$, where each c_α is a unit in A and $\mathbf{Q}^\alpha$ a monomial in $\mathcal{V}(\gamma_\lambda)$.*

An element $(Q,Ex(Q))\in\Lambda(\gamma_\lambda)$ is said to be ***essential at the level*** *γ_λ if Q is not an inessential predecessor of an element of $\Lambda(\gamma_\lambda)$.*

Let $\Psi(\gamma_\lambda)$ be the subset of $\Lambda(\gamma_\lambda)$ consisting of all the essential roots at the level γ_λ. Let $\mathcal{V}(\gamma_\lambda)$ be the subset of $\Psi(\gamma_\lambda)$ consisting of all $(Q,\mathrm{Ex}(Q))$ such that $\mathrm{in}_\nu(Q)$ does not belong to the k-vector space of $\mathrm{gr}_\nu(A)$ generated by the set $\{\mathrm{in}_\nu\mathbf{Q}^\gamma\}$ where $\mathbf{Q}^\gamma$ runs over the set of all the generalized monomials on roots preceding Q in the above ordering.
We extend the well ordering from $\Lambda(<\gamma_\lambda)$ to $\Lambda(\gamma_\lambda)$ by postulating that $\Lambda(<\gamma_\lambda)$ is the initial segment of $\Lambda(\gamma_\lambda)$. Moreover, we extend this well ordering from $\Lambda(\gamma_\lambda)$ to $\Lambda(\gamma_\lambda)\cup\Theta(\gamma_{\lambda-1})$.

If ℓ is not a limit ordinal, let $E(\ell)=\mathrm{In}(\ell)+\mathbb{N}^{\mathcal{V}(\gamma_\ell)}\subset\mathbb{N}^{\mathcal{V}(\gamma_\ell)}$. Now, if $\ell'<\ell''$, we have $\mathcal{V}(\gamma_{\ell'})\subset\mathcal{V}(\lambda_{\ell''})$, which induces an inclusion $\mathbb{N}^{\mathcal{V}(\gamma_{\ell'})}\subset\mathbb{N}^{\mathcal{V}(\gamma_{\ell''})}$. If ℓ is a limit ordinal, define $E(\ell)=\bigcup\limits_{\ell'<\ell}E(\ell')$.

Notation. Denote by $\Theta(<\gamma_\lambda)$ the set $\bigcup\limits_{\ell<\lambda}\Theta(\gamma_\ell)\setminus\Lambda(<\gamma_\lambda)$.

Remark 1.4.5 *We have*

$$\Psi(\gamma_\lambda)\cup\Theta(<\gamma_\lambda)=\Psi(<\gamma_\lambda)\cup\Theta(<\gamma_\lambda). \tag{50}$$

Indeed, consider an element $(Q,Ex(Q))\in\Psi(\gamma_\lambda)\cup\Theta(<\gamma_\lambda)$. If λ is a limit ordinal, then

$$(Q,Ex(Q))\in\Psi(<\gamma_\lambda)\cup\Theta(<\gamma_\lambda) \tag{51}$$

by (43). If λ is not a limit ordinal and $(Q, Ex(Q)) \in \Psi(\gamma_\lambda) \setminus \Psi(\gamma_{\lambda-1})$ then

$$(Q, Ex(Q)) \in \Theta(\gamma_{\lambda-1})$$

by (49). Thus (51) holds in all the cases and (50) is proved.

Lemma 1.4.6 *The set $\mathbf{Q}(h) = \left\{ (Q, Ex(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(<\gamma_\lambda) \mid Q \notin \mathfrak{m}^h \right\}$ is finite for every $h \in \mathbb{N}$.*

Proof: Consider an element $(Q, \mathrm{Ex}(Q)) \in \mathbf{Q}(h)$. If $(Q, Ex(Q)) \in \Theta(<\gamma_\lambda)$, then there exists $\ell < \lambda$ such that $(Q, Ex(Q)) \in \Theta(<\gamma_\ell)$. If $(Q, Ex(Q)) \in \Psi(<\gamma_\lambda) \subset \Lambda(\gamma_\lambda) = \bigcup_{\ell<\lambda} \Lambda(\gamma_\ell)$, then there exists $\ell < \lambda$ such that $(Q, Ex(Q)) \in \Lambda(\gamma_\ell)$. Since Q is essential at level γ_λ, it is also essential at level γ_ℓ, so $(Q, Ex(Q)) \in \Psi(\gamma_\ell)$. Thus by the induction hypothesis on λ, for any $Q \in \mathbf{Q}(h)$, we have $\nu_\mathfrak{m}(Q) = \nu_\mathfrak{m}(\mathrm{In}\ Q)$.

Write $\mathrm{Ex}(Q) = \mathbf{Q}^{\alpha_0} + \cdots$ where, by construction, $\mathbf{Q}^{\alpha_0}$ is either a u_r or a product of at least 2 terms, $\mathbf{Q}^{\alpha_0} = \prod Q_s^{\beta_s}$.

In the first case, the number of such $\mathbf{Q}^{\alpha_0}$ is finite, because the number of u_k is finite.

In the second case, $\nu_\mathfrak{m}(Q_s) < \nu_\mathfrak{m}(\mathbf{Q}^{\alpha_0}) \le \nu_\mathfrak{m}(Q) < h$. So $\nu_\mathfrak{m}(Q_s) < h-1$ and, by induction on h, the number of such Q_s is finite. If

$$m = \min \left\{ \nu_\mathfrak{m}(Q_s) \mid Q_s \text{ divides } \mathbf{Q}^{\alpha_0} \right\},$$

then $|\alpha_0| m \le \nu_\mathfrak{m}(\mathbf{Q}^{\alpha_0}) \le h-1$, so there is a finite number of such α_0 possible which means that the number of such $\mathbf{Q}^{\alpha_0}$ is finite. By the induction hypothesis, Q is completely determined by In Q whenever $(Q, \mathrm{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(<\gamma_\lambda)$. Therefore $\mathbf{Q}(h)$ is finite. □

Corollary 1.4.7 *The set of monomials $\{\mathbf{Q}^\alpha \mid \mathbf{Q}^\alpha \notin \mathfrak{m}^s\}$ in $\Psi(\gamma_\lambda) \cup \Theta(<\gamma_\lambda)$ is finite for every $s \in \mathbb{N}$.*

Corollary 1.4.8 *(1) Any infinite sequence of generalized monomials in $\Psi(\gamma_\lambda) \cup \Theta(<\gamma_\lambda)$, all of whose members are distinct, converges to 0 in the* **m***-adic topology.*

(2) Any infinite series, all of whose terms are distinct generalized monomials in $\Psi(\gamma_\lambda) \cup \Theta(<\gamma_\lambda)$ converges in the **m***-adic topology.*

Lemma 1.4.9 *The set*

$$\mathbf{Q}^\alpha = \prod Q^{\alpha_Q} \text{ such that } (Q, Ex(Q)) \in \Psi(\gamma_\lambda) \cup \{(Q, , Ex(Q)) \in \Theta(<\gamma_\lambda) \mid \nu(Q) = \gamma_\lambda\}$$

and $\nu(\mathbf{Q}^\alpha) = \gamma_\lambda$ is finite.

Proof: By the Artin-Rees lemma, there exists p_0 such that, for $p \ge p_0$,

$$\mathfrak{m}^p \cap P_{\gamma_\lambda} = \mathfrak{m}^{p-p_0}(\mathfrak{m}^{p_0} \cap P_{\gamma_\lambda}).$$

Take $p > p_0$, then

$$\mathfrak{m}^p \cap P_{\gamma_\lambda} \subset \mathfrak{m} P_{\gamma_\lambda} \subset P_{\gamma_\lambda +}. \tag{52}$$

This equation shows that the set of the lemma is disjoint from $\mathfrak{m}^p$. So by the above corollary, the set of the lemma is finite. □

Consider now the ordered set $\{\mathbf{Q}^{\alpha_1}, \ldots, \mathbf{Q}^{\alpha_s}\}$ of monomials

$$\mathbf{Q}^\alpha = \prod Q^{\alpha_Q},\ (Q, \mathrm{Ex}(Q)) \in \mathcal{V}(\gamma_\lambda) \cup \{(Q, \mathrm{Ex}(Q)) \in \Theta(<\gamma_\lambda) \mid \nu(Q) = \gamma_\lambda\} \tag{53}$$

of value γ_λ such that the natural projection of α to $\mathbb{N}^{\mathcal{V}(\gamma_\lambda)}$ does not belong to $E(\lambda)$. The fact that this set is **finite** follows from the above Lemma and the fact that $\mathcal{V}(\gamma_\lambda) \subset \Psi(\gamma_\lambda)$.

Let $i_1 = \max\left\{ i \in \{1,\ldots,s\} \;\middle|\; \mathrm{in}_\nu\left(\mathbf{Q}^{\alpha_i}\right) \in \sum_{j=i+1}^{s} k\ \mathrm{in}_\nu\left(\mathbf{Q}^{\alpha_j}\right) \right\}$ and consider the unique relation $\mathrm{in}_\nu\left(\mathbf{Q}^{\alpha_{i_1}}\right) - \sum_{j=i_1+1}^{s} c_{1j}\ \mathrm{in}_\nu\left(\mathbf{Q}^{\alpha_j}\right) = 0$. Let $P_1 = \mathbf{Q}^{\alpha_{i_1}} - \sum_{j=i_1+1}^{s} c_{1j}\mathbf{Q}^{\alpha_j}$ where we view k as a subring of A via the identification (41).

Let $i_2 = \max\left\{ i \in \{1,\ldots,i_1-1\} \;\middle|\; \mathrm{in}_\nu\left(\mathbf{Q}^{\alpha_i}\right) \in \sum_{j=i+1}^{s} k\ \mathrm{in}_\nu\left(\mathbf{Q}^{\alpha_j}\right) \right\}$ and, as before, consider the unique $P_2 = \mathbf{Q}^{\alpha_{i_2}} - \sum\limits_{\substack{j=i_2+1 \\ j\neq i_1}}^{s} c_{2j}\mathbf{Q}^{\alpha_j}$ such that the vector $(\alpha_j)_{j=i_1+1,\ldots,s},\ c_{2j} \neq 0$, is minimal in the lexicographical order and define so on uniquely $P_3,\ldots,P_t$.

Now, if λ has a predecessor, we let

$$\Theta(\gamma_\lambda) = \{\,(Q,\mathrm{Ex}(Q)) \in \Theta(<\gamma_\lambda)\ |\ \nu(Q) \geq \gamma_\lambda\} \cup \{(P_1,\mathrm{Ex}(P_1)),\ldots,(P_t,\mathrm{Ex}(P_t))\} \tag{54}$$

where

$$\mathrm{Ex}(P_j) = \mathbf{Q}^{\alpha_{i_j}} - \sum_k c_{jk}\mathbf{Q}^{\alpha_k} \tag{55}$$

if $\mathbf{Q}^{\alpha_{i_j}}$ is not a preceding root Q and

$$\mathrm{Ex}(P_j) = \mathrm{Ex}(Q) - \sum_k c_{jk}\mathbf{Q}^{\alpha_k} \tag{56}$$

in the other case. We define the order on $\Theta(\gamma_\lambda)$ by $\Theta(\gamma_{\lambda-1}) < \{(P_1,\mathrm{Ex}(P_1)),\ldots,(P_t,\mathrm{Ex}(P_t))\}$ and $(P_1,\mathrm{Ex}(P_1)) < \cdots < (P_t,\mathrm{Ex}(P_t))$.

Remark 1.4.10 *Note that, because the system of coordinates is prepared, $u_1,\ldots,u_n$ are always essential.*

Remark 1.4.11 *Note that Remark 1.2.8 remains valid in this context, with the obvious modification that the expressions of approximate roots are now allowed to be infinite, but convergent in the $\mathfrak{m}$-adic topology.*

Suppose now λ is a limit ordinal. Let $(Q_0, Ex(Q_0)) \in \Lambda(\gamma_{\ell_0})$ for some $\ell_0 < \lambda$ and $\mathbf{Q}^\alpha = In(Q_0)$. Let $L(Q_0)$ be the following infinite well ordered set of approximate roots, indexed by ordinals ℓ, $\ell_0 \leq \ell < \lambda$

$$L(Q_0) = \{\ (Q^{(\ell)}, Ex(Q^{(\ell)})) \in \Psi(\gamma_\ell)\ \}_{\ell_0\leq\ell<\lambda}$$

such that $\mathrm{In}Q^{(\ell)} = \mathbf{Q}^\alpha$.

By Remarks 1.2.8 and 1.4.11, for $\ell_0 \leq \ell < \ell' < \lambda$, we have

$$\mathrm{Ex}(Q^{(\ell')}) = \mathrm{Ex}(Q^{(\ell)}) + \sum_{j\in W} c_j\mathbf{Q}^{\alpha_j} \tag{57}$$

where $\nu(\mathbf{Q}^{\alpha_j}) \geq \nu(Q^{(\ell)})$.

Let p be a positive integer. By induction assumption, all the approximate roots Q appearing in any of the monomials $\mathbf{Q}^{\alpha_j}$ belong to $\mathcal{V}(\gamma_\lambda)$ and, by lemma 1.4.6, the number of such roots outside $\mathfrak{m}^p$ is finite. Thus, all but finitely many $\mathbf{Q}^{\alpha_j}$ belong to $\mathfrak{m}^p$. This proves that $L(Q_0)$ has a limit in A with respect to the $\mathfrak{m}$-adic topology : $(\lim\limits_{\to} Q, \lim\limits_{\to}\mathrm{Ex}(Q))$.

Let

$$\Theta(\gamma_\lambda) = \{(Q,\mathrm{Ex}(Q)) \in \Theta(<\gamma_\lambda)\ |\ \nu(Q) \geq \gamma_\lambda\} \cup \hat{L} \tag{58}$$

where $\hat{L}$ consists of all couples of the form $(\lim\limits_{\to} Q, \lim\limits_{\to}\mathrm{Ex}(Q))$.

So finally, the expression of an approximate root has the form

$$\mathrm{Ex}(Q) = \mathbf{Q}^{\alpha} + \sum_k a_k \mathbf{Q}^{\alpha_k} \tag{59}$$

the sum, written in the increasing order of the monomials, being finite or infinite.

We now prove the finiteness of sets (46) and (47). First, note that the set

$$\{(Q, \mathrm{Ex}(Q)) \in \Theta(< \gamma_\lambda) \cup \Lambda(\gamma_\lambda) \mid \nu(Q) = \gamma\},\ \gamma \in \Phi \tag{60}$$

is finite by the induction hypothesis and the set

$$\{(Q, \mathrm{Ex}(Q)) \in \Psi(\gamma_\lambda) \cup \Theta(< \gamma_\lambda) \mid Q \notin \mathfrak{m}^p\},\ p \in \mathbb{N} \tag{61}$$

is finite by the induction hypothesis and lemma (1.4.9). If λ is not a limit ordinal, the finiteness of (46) and (47) follows from the fact that the set $\Theta(\gamma_\lambda) \setminus \Theta(< \gamma_\lambda)$ is finite by construction. If λ is a limit ordinal, to prove finiteness of (46) and (47), it remains to prove that the set

$$\{(Q, \mathrm{Ex}(Q)) \in \hat{L} \mid \nu(Q) = \gamma\} \tag{62}$$

is finite. This is proved in exactly the same way as lemma (1.4.9). This completes the proof of the finiteness of (46) and (47).

The property that the monomials appearing in $\mathrm{Ex}(Q)$ are arranged in increasing order with respect to the ν-adic value holds for all the newly constructed approximate roots. Next we show that $\nu_{\mathfrak{m}}(\mathrm{In}Q) = \nu_{\mathfrak{m}}(Q)$ for all those new approximate roots. Indeed, if λ is not a limit ordinal and $\mathrm{Ex}(Q)$ is given by formula (55), all the monomials appearing in $\mathrm{Ex}(Q)$ have the same ν-adic value and their $\nu_{\mathfrak{m}}$-adic values are increasing because of the order we imposed on monomials which proves that $\nu_{\mathfrak{m}}(\mathrm{In}Q) = \nu_{\mathfrak{m}}(Q)$. If $(Q', \mathrm{Ex}(Q'))$ is an approximate root whose expression is given by formula (56), with P_j playing the role of Q', let $\mathbf{Q}^{\alpha_0} = \mathrm{In}\ Q$. We have $Q' = Q + \sum c_\alpha \mathbf{Q}^{\alpha}$, where $\nu(\mathbf{Q}^{\alpha}) = \nu(Q)$. Then $\nu_{\mathfrak{m}}(Q) \le \nu_{\mathfrak{m}}(\mathbf{Q}^{\alpha})$ for all α, because of the order on monomials. So that finally, $\nu_{\mathfrak{m}}(\mathbf{Q}^{\alpha_0}) \le \nu_{\mathfrak{m}}(Q) \le \nu_{\mathfrak{m}}(\mathbf{Q}^{\alpha})$, which proves that $\nu_{\mathfrak{m}}(\mathrm{In}Q') = \nu_{\mathfrak{m}}(Q')$. The property that $\nu_{\mathfrak{m}}(\mathrm{In}Q) = \nu_{\mathfrak{m}}(Q)$ is clearly preserved by passing to the limit, so it also holds in the case when λ is a limit ordinal.

Remark 1.4.12 *We just showed that there is a one to one correspondence between the approximate roots $Q \in \Psi(\gamma_\ell)$ and the set of monomials which are the first term of the expression $Ex(Q)$ of such an approximate root Q. Let us denote by $\mathbf{M}(\ell)$ the set of those monomials.*

We well order $\hat{L}$ by the lexicographical order of the triples $(\nu(Q), \nu_{\mathfrak{m}}(Q), \mathrm{In}(Q))$,$Q \in \hat{L}$. We extend this ordering to $\Theta(\gamma_\lambda)$ by postulating that $\hat{L}$ is the final segment in $\Theta(\gamma_\lambda)$.

The rest of Theorem 1.4.3 holds by construction.

1.5 Standard form in the case of complete regular local rings

Let $\Psi(\gamma_\Phi) = \bigcup_{\ell<\Phi} \bigcap_{\ell \le \ell' < \Phi} \Psi(\gamma_{\ell'})$ and let $\mathcal{V}(\gamma_\Phi)$ be the set of approximate roots, essential at the level γ_Φ.

In this section, we fix an ordinal $\lambda \le \Phi$.

Definition 1.5.1 *A monomial in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$ is called standard with respect to λ if all the approximate roots appearing in it belong to $\mathcal{V}(\gamma_\lambda)$ and it is not divisible by any InQ where Q is an approximate root in $(\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \ldots, (u_n, u_n)\}$.*

Take an ordinal $\ell \le \lambda$.

Definition 1.5.2 *Let $f \in A$. An expansion of f of the form $f = \sum c_\alpha \mathbf{Q}^{\alpha}$ where the $\mathbf{Q}^{\alpha}$ are monomials in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$, written in increasing order, is a standard form of level γ_ℓ if $\forall \gamma' < \gamma_\ell$ and for all α such that $\nu(\mathbf{Q}^{\alpha}) = \gamma'$, $\mathbf{Q}^{\alpha}$ is a standard monomial.*

We now construct by induction on ℓ a standard form of f of level γ_ℓ. We will write this standard form as

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

where, for all α, $\mathbf{Q}^\alpha$ is a generalized monomial in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$, $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$ and f_ℓ is a sum of standard monomials in $\mathcal{V}(\gamma_\lambda)$ of value strictly less than γ_ℓ.

To start the induction, let $f_0 = 0$. The *standard form of f of level 0* will be its expansion, $f = f_0 + \sum c_\alpha \mathbf{u}^\alpha$, written in increasing order according to the monomial order defined above, as a formal power series in the u_i.

Let $\ell < \lambda$ be an ordinal. Let us define $f_{\ell+1}$ and the standard form of f of level $\gamma_{\ell+1}$ as follows. Assume, inductively, that a standard form of level γ_ℓ is already defined: $f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$ with $\nu(\mathbf{Q}^\alpha) \geq \gamma_\ell$, for all α, and the value of any monomial of f_ℓ is strictly less than γ_ℓ.

Take the homogeneous part of $\sum c_\alpha \mathbf{Q}^\alpha$ of value γ_ℓ, the monomials being written in increasing order. Assume that not all the $\mathbf{Q}^\alpha$ are standard with respect to λ, and take the smallest non standard $\mathbf{Q}^\alpha$. Since $\mathbf{Q}^\alpha$ is not standard, one of the two following conditions holds:

1. There exists an approximate root $Q \in (\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)) \setminus \{(u_1, u_1), \ldots, (u_n, u_n)\}$ such that $In(Q)$ divides $\mathbf{Q}^\alpha$. Write $Q = In(Q) + \sum c_\beta \mathbf{Q}^\beta$ and replace $In(Q)$ by $Q - \sum c_\beta \mathbf{Q}^\beta$ in $\mathbf{Q}^\alpha$.
2. An approximate root $Q \in \Psi(\gamma_\lambda) \setminus \mathcal{V}(\gamma_\lambda)$ divides $\mathbf{Q}^\alpha$. Since $Q \notin \mathcal{V}(\gamma_\lambda)$, there exists

   $$Q' \in \Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$$

   of the form $Q' = Q + \sum_\beta d_\beta \mathbf{Q}^\beta$, where the $\mathbf{Q}^\beta$ are monomials in $\mathcal{V}(\gamma_\lambda)$ of value greater than or equal to γ_ℓ. Replace Q by $Q' - \sum_\beta d_\beta \mathbf{Q}^\beta$.

In both cases, those changes introduce new monomials, with increasing $\nu_{\mathfrak{m}}$ value, but either they are of value strictly greater than γ_ℓ or they are of value exactly γ_ℓ but greater than $\mathbf{Q}^\alpha$ in the monomial ordering. We repeat this procedure as many times as we can. After a finite number of steps, no more changes are available involving monomials of value exactly γ_ℓ. Then, let $f_{\ell+1} = f_\ell + \sum d_\rho \mathbf{Q}^\rho$ with $\nu(\mathbf{Q}^\rho) = \gamma_\ell$, so that $f = f_{\ell+1} + \sum c_\alpha \mathbf{Q}^\alpha$ where $\nu(\mathbf{Q}^\alpha) > \gamma_\ell$.

Suppose now that μ is a limit ordinal. For each $\ell < \mu$, write $f = f_\ell + \delta_\ell$ where f_ℓ is a sum of standard monomials, with respect to λ, of value strictly less than γ_ℓ and δ_ℓ is a sum of monomials in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$, of value greater than or equal to γ_ℓ. We assume inductively that, for each $\ell < \mu$ and for each generalized monomial $\mathbf{Q}^\tau$ in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$, there exist $c_\tau, b_\tau \in k$ and an ordinal $\ell_0 < \ell$ such that, for all ℓ', $\ell_0 < \ell' < \ell$, the monomial $\mathbf{Q}^\tau$ appears in $f_{\ell'}$ with coefficient c_τ and in $\delta_{\ell'}$ with coefficient b_τ. Moreover, assume that $f_\ell = \lim\limits_{\overrightarrow{\ell' < \ell}} f_{\ell'} = \sum_\tau c_\tau \mathbf{Q}^\tau$ and $\delta_\ell = \lim\limits_{\overrightarrow{\ell' < \ell}} \delta_{\ell'} = \sum_\tau b_\tau \mathbf{Q}^\tau$.

Lemma 1.5.3 *Consider a generalized monomial $\mathbf{Q}^\tau$ in $\Psi(\gamma_\lambda) \cup \Theta(\gamma_\lambda)$. There exist $c_\tau, b_\tau \in k$ and an ordinal $\ell_0 < \mu$ such that, for all ℓ, $\ell_0 < \ell < \mu$, the monomial $\mathbf{Q}^\tau$ appears in f_ℓ with coefficient c_τ and in δ_ℓ with coefficient b_τ.*

Corollary 1.5.4 *The limits $\lim\limits_{\overrightarrow{\ell < \mu}} f_\ell$ and $\lim\limits_{\overrightarrow{\ell < \mu}} \delta_\ell$ exist in the $\mathfrak{m}$-adic topology.*

Proof of Corollary 1.5.4 : This is an immediate consequence of the Lemma and Corollary 1.4.8. $\square$

Proof of Lemma 1.5.3: The existence of c_τ in the lemma follows immediately from the construction and the induction hypothesis.

If $\nu(\mathbf{Q}^\tau) < \gamma_\mu$, put $b_\tau = 0$. Assume $\nu(\mathbf{Q}^\tau) \geq \gamma_\mu$. For $\ell < \mu$, let $b_\tau(\ell)$ denote the coefficient of $\mathbf{Q}^\tau$ in δ_ℓ. Take an ordinal $\ell < \mu$. Suppose

$$b_\tau(\ell) \neq b_\tau(\ell+1). \tag{63}$$

This means that in the above construction of $f_{\ell+1}+\delta_{\ell+1}$ from $f_\ell+\delta_\ell$, $\mathbf{Q}^\tau$ appears in one of the expressions $\frac{\mathbf{Q}^\alpha}{InQ}Q, \frac{\mathbf{Q}^\alpha}{InQ}\sum\limits_\beta d_\beta \mathbf{Q}^\beta$ (case 1 of the construction) or $\frac{\mathbf{Q}^\alpha}{Q}Q'$, $\frac{\mathbf{Q}^\alpha}{Q}\sum\limits_\beta d_\beta \mathbf{Q}^\beta$ (case 2 of the construction). Then

$$\nu_{\mathfrak{m}}(\mathbf{Q}^\alpha) \leq \nu_{\mathfrak{m}}(\mathbf{Q}^\tau). \tag{64}$$

Suppose that there were infinitely many ℓ for which (63) holds. This would mean that there are infinitely many monomials $\mathbf{Q}^\alpha$ (all distinct because $\nu(\mathbf{Q}^\alpha) = \gamma_\ell$), satisfying (64). This contradicts Lemma 1.4.6; hence there are finitely many such ℓ. Together with the induction hypothesis, this proves that $b_\tau(\ell)$ stabilizes for ℓ sufficiently large. This completes the proof of the lemma. □

For each $\mathbf{Q}^\tau$ as above, let c_τ, b_τ be as in Lemma 1.5.3. Let $f_\mu = \lim\limits_{\overrightarrow{\ell<\mu}} f_\ell = \sum_\tau c_\tau \mathbf{Q}^\tau$ and $\delta_\mu = \lim\limits_{\overrightarrow{\ell<\mu}} \delta_\ell = \sum_\tau b_\tau \mathbf{Q}^\tau$. We define the standard form of f of level γ_μ as $f = f_\mu + \delta_\mu$.

This completes the construction of standard form of level γ_ℓ for $\ell \leq \lambda$.

Proposition 1.5.5 *Let*

$$f = f_\ell + \sum c_\alpha \mathbf{Q}^\alpha$$

be a standard form of f of level γ_ℓ and $\gamma < \gamma_\ell$ an element of Φ. Then $\sum\limits_{\nu(\mathbf{Q}^\beta)=\gamma} c_\beta \mathbf{Q}^\beta \notin P_{\gamma+}$.

The proof is entirely the same as the proof of the analogous Proposition 1.3.3.

For each ℓ, the part f_ℓ of a standard form of f of level γ_ℓ is uniquely determined. This is a straightforward consequence of the proposition.

By Proposition 1.5.5, if $\gamma_\ell > \nu(f)$ then $\nu(f)$ equals the smallest value of a monomial appearing in the standard form of f of level γ_ℓ.

Theorem 1.5.6 *(1) Take $\gamma \in \Phi$, $\gamma < \gamma_\lambda$. Then $\frac{P_\gamma}{P_{\gamma+}}$ is generated as a k-vector space by $\{in_\nu \mathbf{Q}^\beta\}$ where $\mathbf{Q}^\beta$ runs over the set of all standard monomials with respect to λ, satisfying $\nu(\mathbf{Q}^\beta) = \gamma$.*

(2) The part of the graded k-algebra $gr_\nu(A)$ of degree strictly less than γ_λ is generated by the initial forms of the approximate roots of $\mathcal{V}(\gamma_\lambda)$.

The same proof as that of Theorem 1.3.4 works here.

Now, for each ordinal ℓ, let $X = X_{\mathcal{V}(\gamma_\ell)}$ be a set of independent variables, indexed by $\mathcal{V}(\gamma_\ell)$ and consider the graded k-algebra $k\left[X_{\mathcal{V}(\gamma_\ell)}\right]$, where we define deg $X_j = \nu(Q_j)$. Let P denote the homogeneous monomial ideal of $k\left[X_{\mathcal{V}(\gamma_\ell)}\right]$ generated by all the monomials in $X_{\mathcal{V}(\gamma_\ell)}$ of degree greater than or equal to $\gamma_{\ell+1}$. We have the natural map

$$\phi_\ell : \begin{array}{ccc} \frac{k\left[X_{\mathcal{V}(\gamma_\ell)}\right]}{P} & \to & \frac{\mathrm{gr}_\nu A}{P_{\gamma_{\ell+1}}} \\ X_j & \mapsto & \mathrm{in}_\nu Q_j \end{array}.$$

Now, for $\ell = 0$, let $I_0 = (0)$. For $\ell > 0$, let I_ℓ denote the ideal of $\frac{k\left[X_{\mathcal{V}(\gamma_\ell)}\right]}{P}$ generated by $I_{<\ell}$ and all the homogeneous polynomials of the form

$$X^{\alpha_0} + \lambda_1 X^{\alpha_1} + \lambda_2 X^{\alpha_2} + \cdots + \lambda_{b_0} X^{\alpha_{b_0}} \tag{65}$$

where $\mathbf{Q}^{\alpha_0}+\lambda_1\mathbf{Q}^{\alpha_1}+\lambda_2\mathbf{Q}^{\alpha_2}+\dots+\lambda_{b_0}\mathbf{Q}^{\alpha_{b_0}}$ is the homogeneous part of least degree of $Ex(Q)$, $Q\in\mathcal{V}(\gamma_\ell)\cup\Theta(\gamma_\ell)$.

Once again the proofs of Corollary 1.3.5 and Corollary 1.3.6 give the analogous corollaries :

Corollary 1.5.7 *We have $Ker\ \phi_\ell=I_\ell$.*

Corollary 1.5.8 *Take an element $\gamma\in\Phi$, $\gamma<\gamma_\lambda$. The valuation ideal P_γ is generated by all the generalized monomials of value $\geq\gamma$ in $\{Q\mid(Q,Ex(Q))\in\Psi(\gamma_\lambda)\}$. The ideal P_{γ_λ} is generated by all the generalized monomials of value $\geq\gamma_\lambda$ in $\{Q\mid(Q,Ex(Q))\in\Psi(\gamma_\lambda)\cup\Theta(\gamma_\lambda)\}$.*

Part 2. Separating ideal and connectedness

2.1 A description of the separating ideal.

Let A be a noetherian ring and α and β points in Sper A. The purpose of this section is twofold. First we prove a general result on the behaviour of $<\alpha,\beta>$ under localization. Secondly, we restrict attention to the case when A is regular and is either complete or $<\alpha,\beta>$ is primary to a maximal ideal of A. In this case, we describe generators of the separating ideal $<\alpha,\beta>$ as generalized monomials in those approximate roots Q_j which are common to ν_α and ν_β.

We will need the following basic properties of the separating ideal, proved in [26]:

Proposition 2.1.1 *Let the notation be as above. We have:*

(1) $<\alpha,\beta>$ is both a ν_α-ideal and a ν_β-ideal.

(2) α and β induce the same ordering on $\frac{A}{<\alpha,\beta>}$ (in particular, the set of ν_α-ideals containing $<\alpha,\beta>$ coincides with the set of ν_β-ideals containing $<\alpha,\beta>$).

(3) $<\alpha,\beta>$ is the smallest ideal (in the sense of inclusion), satisfying (1) and (2).

(4) If α and β have no common specialization then $<\alpha,\beta>=A$.

Notation. If $\mathfrak{p}\in$ Sper A, $\mathfrak{p}_\alpha\subset\mathfrak{p}$, the notation $\alpha A_\mathfrak{p}$ will stand for the point of Sper $A_\mathfrak{p}$ with support $\mathfrak{p}_\alpha A_\mathfrak{p}$ and the total order on $\dfrac{A_\mathfrak{p}}{\mathfrak{p}_\alpha A_\mathfrak{p}}$ given by $\leq_\alpha$.

Proposition 2.1.2 *Let A be a ring. Consider points $\alpha,\beta\in SperA$ whose respective supports are $\mathfrak{p}_\alpha,\mathfrak{p}_\beta$ and let ϵ be a common specialization of α and β with support $\mathfrak{p}$.*

(1) We have $<\alpha,\beta>A_\mathfrak{p}=<\alpha A_\mathfrak{p},\beta A_\mathfrak{p}>$.

(2) Let $\mathfrak{p}$ be a prime ideal of A, containing $<\alpha,\beta>$. Then

$$<\alpha,\beta>\subset<\alpha,\beta>A_\mathfrak{p}\cap A. \tag{66}$$

with equality if $<\alpha,\beta>$ is $\mathfrak{p}$-primary.

(3) If $\mathfrak{p}=\mathfrak{p}_\epsilon$ with ϵ the unique common specialization of α and β (in particular, whenever

$$\mathfrak{p}=\sqrt{<\alpha,\beta>}$$

and $\mathfrak{p}$ is maximal), we have equality in (66).

Remark 2.1.3 *In (2) of the Proposition, the special case of interest for applications is $\mathfrak{p}=\mathfrak{p}_\epsilon$, with $\epsilon\in$ Sper A a common specialization of α and β.*

Proof: Let f be a generator of $<\alpha,\beta>$ such that f changes sign between α and β. Say, $f(\alpha)\geq0$ and $f(\beta)\leq0$. As the orders on $A/\mathfrak{p}_\alpha$ and $A_\mathfrak{p}/\mathfrak{p}_\alpha A_\mathfrak{p}$ are the same (the quotient field is the same) — and similarly for $\mathfrak{p}_\beta$ — f changes sign between $\alpha A_\mathfrak{p}$ and $\beta A_\mathfrak{p}$. Thus $f\in<\alpha A_\mathfrak{p},\beta A_\mathfrak{p}>$.

Conversely, a generator of $< \alpha A_{\mathfrak{p}}, \beta A_{\mathfrak{p}} >$ is of the form g/s, $s \notin \mathfrak{p}$, such that $\frac{g}{s}(\alpha A_{\mathfrak{p}}) \geq 0$ and $\frac{g}{s}(\beta A_{\mathfrak{p}}) \leq 0$, for instance. But, as $\mathfrak{p}$ is a specialisation of α and β and $s \notin \mathfrak{p}$, s has the same sign on α and β (and is non-zero at both points), so g keeps different signs on α and β which means that $g \in < \alpha, \beta >$, and, consequently, $\frac{g}{s} \in < \alpha, \beta > A_{\mathfrak{p}}$. This proves (1) of the Proposition.

(2) of the Proposition is a standard general statement about localization of ideals at a prime ideal.

(3) of the Proposition follows immediately from the fact that $\mathfrak{p}$ is the center of the valuation ν_α and $< \alpha, \beta >$ is a ν_α-ideal. $\Box$

Let $(A, \mathfrak{m}, k)$ be a regular local ring and α and β two points of $\mathrm{Sper}(A)$ having a common specialization ϵ whose center is the maximal ideal $\mathfrak{m}$ of A. Then ν_α and ν_β are both centered at $\mathfrak{m}$.

Let $\Phi_\alpha = \nu_\alpha(A \setminus \{0\})$ and $\Phi_\beta = \nu_\beta(A \setminus \{0\})$. Let $\gamma_{\alpha s}$ be the s-th element of Φ_α and similarly for β. Let $P_{\gamma_{\alpha s}}$ denote the ν_α-ideal of value $\gamma_{\alpha s}$ and similarly for $P_{\gamma_{\beta s}}$. Let r be the ordinal such that $\gamma_{\alpha r} = \nu_\alpha(< \alpha, \beta >)$. Then $\gamma_{\beta r} = \nu_\beta(< \alpha, \beta >)$ by Proposition 2.1.1. We have $P_{\gamma_{\alpha s}} = P_{\gamma_{\beta s}}$ for $s = 1, \ldots, r$ by Proposition 2.1.1.

Let $Q_j(\alpha)$ denote the j-th approximate root for ν_α (in the case when A is complete j is an ordinal rather than a natural number); we will denote the monomials in these approximate roots by $\mathbf{Q}(\alpha)^\gamma$; similarly for $Q_j(\beta)$ and $\mathbf{Q}(\beta)^\gamma$. Let us consider the sequences of vectors $\mathbf{m}_i = (m_{i1}, m_{i2}, \ldots, m_{it_{i\alpha}})$, $m_{ij} \in P_{\gamma_{\alpha i}}/P_{\gamma_{\alpha, i+1}}$ which are the initial forms of the monomials $\mathbf{Q}(\alpha)^{\alpha_{ij}}$ of value $\gamma_{\alpha i}$ (see section 1.2 and (53)). We do the same with ν_β and write $\mathbf{n}_1, \mathbf{n}_2, \ldots$ the corresponding sequences of initial forms.

Let $M_{\alpha h}$ be the set of all the generalized monomials in $\mathbf{Q}(\alpha)$, of value $\gamma_{\alpha h}$ with respect to ν_α. Let $M_{\beta h}$ be the same kind of set with respect to ν_β. Now, let $s_{\alpha h}$ denote the cardinality of $M_{\alpha h}$; similarly for $s_{\beta h}$.

For a given ℓ, consider the following three conditions $(1)_\ell$, $(2)_\ell$, $(3)_\ell$:

$(1)_\ell$ $s_{\alpha i} = s_{\beta i}$, $1 \leq i \leq \ell$

$(2)_\ell$ $M_{\alpha i} = M_{\beta i}$ for $i \leq \ell$

$(3)_\ell$ For any $i \leq \ell$ and $\bar{\lambda}_1, \ldots, \bar{\lambda}_{s_{\alpha i}} \in k$, the sign on α of the linear combination $\sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j m_{ij}$ is the same as the sign on β of $\sum_{j=1}^{s_{\alpha i}} \bar{\lambda}_j n_{ij}$ (here we adopt the convention that the sign can be strictly positive, strictly negative or zero) where m_{ij}, n_{ij} are the initial forms of the monomials $\mathbf{Q}(\alpha)^{\alpha_{ij}}, \mathbf{Q}(\beta)^{\alpha_{ij}}$ in the graded rings $\mathrm{gr}_{\nu_\alpha}(A)$, $\mathrm{gr}_{\nu_\beta}(A)$. Note that if conditions $(1)_\ell$–$(3)_\ell$ hold then the set of k-linear relations among the m_{ij}, $i \leq \ell$, is the same as the set of k-linear relations among the n_{ij}.

Proposition 2.1.4 *The ordinal r is the smallest ordinal r' such that at least one of the conditions $(1)_{r'}$–$(3)_{r'}$ does not hold.*

Proof: Let r' be the smallest ordinal such that at least one of the conditions $(1)_{r'}$–$(3)_{r'}$ does not hold. By definitions, we have $M_{\alpha r'} \neq \emptyset$ and $M_{\beta r'} \neq \emptyset$. We have the following 2 possibilities:

First, suppose $M_{\alpha r'} \neq M_{\beta r'}$ (which includes the case $s_{\alpha r'} \neq s_{\beta r'}$). Say, $M_{\alpha r'} \not\subset M_{\beta r'}$. Take generalized monomials $\mathbf{Q}^\gamma \in M_{\alpha r'} \setminus M_{\beta r'}$, and $\mathbf{Q}^\delta \in M_{\beta r'}$. Then $\nu_\alpha(\mathbf{Q}^\gamma) \leq \nu_\alpha(\mathbf{Q}^\delta)$, but $\nu_\beta(\mathbf{Q}^\gamma) > \nu_\beta(\mathbf{Q}^\delta)$.

Then there exists a linear combination, with coefficients in $(A \setminus \mathfrak{m})$, of $\mathbf{Q}^\gamma$ and $\mathbf{Q}^\delta$, of value $\gamma_{\alpha r'}$ with respect to ν_α, which changes sign between α and β. This shows that

$$\nu_\alpha(< \alpha, \beta >) \leq \gamma_{\alpha r'}$$

in this case.

The second case is $M_{\alpha r'} = M_{\beta r'}$ and there exist $\bar{\lambda}_1, \dots, \bar{\lambda}_{s_{\alpha r'}}$ such that the sign on α of $\sum\limits_{j=1}^{s_{\alpha r'}} \bar{\lambda}_j m_{r'j}$ differs from the sign on β of $\sum\limits_{j=1}^{s_{\alpha r'}} \bar{\lambda}_j n_{r'j}$ (by assumption, we are in the case $s_{\alpha r'} = s_{\beta r'}$). By a small perturbation of the $\bar{\lambda}_j$ (for instance, by adding or subtracting a "small" element of k to $\bar{\lambda}_1$), we can ensure both that $\sum\limits_{j=1}^{s_{\alpha r'}} \bar{\lambda}_j m_{r'j} \neq 0$ in $gr_{\nu_\alpha} A$ and $\sum\limits_{j=1}^{s_{\alpha r'}} \bar{\lambda}_j n_{r'j} \neq 0$ in $gr_{\nu_\beta} A$. But this gives an $f = \sum\limits_{j=1}^{s_{\alpha r'}} \lambda_j \mathbf{Q}^{\alpha_{r'j}} \in A$ which changes signs between α and β. We have $\nu_\alpha(f) = \gamma_{\alpha r'}$ (and $\nu_\beta(f) = \gamma_{\beta r'}$), so $\nu_\alpha(<\alpha, \beta>) \leq \gamma_{\alpha r'}$ also in this case.

Now take an $f \in A$ with $\nu_\alpha(f) < \gamma_{\alpha r'}$. Then $f \in P_{\gamma_{\alpha s}}$,

$$\gamma_{\alpha s} < \gamma_{\alpha r'}, \tag{67}$$

so $\mathrm{in}_{\nu_\alpha}(f) \in P_{\gamma_{\alpha s}}/P_{\gamma_{\alpha s+}}$. By theorem 1.5.6, $\mathrm{in}_{\nu_\alpha}(f)$ is a k-linear combination of $m_{s1}, \dots, m_{st_{s\alpha}}$. By (67) and the definition of r', this linear combination has the same sign for α and for β (in other words, $P_{\gamma_{\alpha s}}/P_{\gamma_{\alpha s+}} = P_{\gamma_{\beta s}}/P_{\gamma_{\beta s+}}$ with same order induced by α and by β. This means that $\mathrm{in}_{\nu_\alpha}(f)$ has the same sign on α and β, so $\nu_\alpha(<\alpha, \beta>) \geq \gamma_{\alpha r'}$). This completes the proof. $\square$

Corollary 2.1.5 *Let $\alpha, \beta \in Sper(A)$, both centered in the maximal ideal. Let r be as above. Denote by $\gamma = \gamma_{\alpha r}$ the ν_α-value of $<\alpha, \beta>$. Let $Q_1, \dots, Q_q$ be the common approximate roots of the valuations ν_α and ν_β. Then $<\alpha, \beta>$ is generated by the generalized monomials in $Q_1, \dots, Q_q$ of ν_α-value $\geq \gamma$ (and the same with ν_β instead of ν_α).*

Proof: As $<\alpha, \beta>$ is a ν_α-ideal (and a ν_β-ideal), this is a consequence of Corollary 1.5.8.

Definition 2.1.6 *For a graded algebra G, we define*

$$G^* = \left\{ \frac{f}{g} \;\middle|\; f, g \in G, g \neq 0 \text{ and homogeneous} \right\} / \sim.$$

where $\dfrac{f}{g} \sim \dfrac{f'}{g'}$ whenever $fg' = f'g$.

The Alvis–Johnston–Madden example. Let us consider α and β in Sper$(R[x,y,z])$ given by curvettes

$$\begin{aligned} x(t) &= t^6, &(68)\\ y(t) &= t^{10} + ut^{11}, &(69)\\ z(t) &= t^{14} + t^{15} &(70) \end{aligned}$$

where u takes 2 distinct values $u_\alpha > 2$ and $u_\beta > 2$. Applying the above procedure, we show that $\nu_\alpha(<\alpha, \beta>) = 31$.
Indeed, we have $Q_1 = x, Q_2 = y, Q_3 = z$ for α and β. The first level approximate roots are

$$\begin{aligned} Q_4 &= y^2 - xz = (2u-1)t^{21} + u^2 t^{22}, &(71)\\ Q_5 &= yz - x^4 = (u+1)t^{25} + ut^{26}, &(72)\\ Q_6 &= z^2 - x^3 y = (2-u)t^{29} + t^{30} &(73) \end{aligned}$$

for both α and β. Let T denote the preimage of $\mathrm{in}_v t$ under the natural map

$$(gr_{\nu_\alpha} R[x,y,z])^* \hookrightarrow (\mathrm{gr}_v R[[t]])^*,$$

so that

$$(gr_{\nu_\alpha} R[x,y,z])^* \cong (R[T])^*.$$

Then $\mathrm{in}_{\nu_\alpha}(yQ_4) = (2u_\alpha - 1)T^{31}$ and $\mathrm{in}_{\nu_\alpha}(xQ_5) = (u_\alpha + 1)T^{31}$, and similarly for β. Since $u_\alpha \neq u_\beta$, the matrix

$$\begin{pmatrix} (2u_\alpha - 1) & (u_\alpha + 1) \\ (2u_\beta - 1) & (u_\beta - 1) \end{pmatrix}$$

is non-singular, so there exists an R-linear combination of $\mathrm{in}_{\nu_\alpha}(yQ_4)$ and $\mathrm{in}_{\nu_\alpha}(xQ_5)$ which is strictly positive on α and strictly negative on β. According to Proposition 2.1.4,

$$\nu_\alpha(< \alpha, \beta >) \leq 31.$$

One can check that 31 is the lowest value for which either there is a linear combination of generalized monomials with this property or the set of monomials of that value for α does not equal the corresponding set for β, so that in fact $\nu_\alpha(< \alpha, \beta >) = 31$.

For the next approximate root

$$Q_7 = yQ_4 + \frac{2u-1}{u+1} Q_5, \tag{74}$$

we have $Q_7(\alpha) \neq Q_7(\beta)$.

2.2 Some sets which are conjecturally connected

Let $(A, \mathfrak{m}, k)$ be a regular local ring. Take $\alpha, \beta \in \mathrm{Sper}A$, both centered at $\mathfrak{m}$, and elements $f_1, \dots, f_r \in A \setminus < \alpha, \beta >$. The Connectedness Conjecture 0.1.11 asserts that there exists a connected set C, containing α, β, such that C is disjoint from the zero set of $f_1 \cdots f_r$.

Assume that either A is complete or $\sqrt{< \alpha, \beta >} = \mathfrak{m}$.

In this section, we describe a set C, which contains α, β, disjoint from the set $f_1 \cdots f_r = 0$, and which we conjecture to be connected. Under the above assumptions, this reduces the Connectedness Conjecture for α and β to proving the connectedness of C.

Let $\mathbf{Q}_\Lambda = \{Q_\lambda, \lambda \in \Lambda\}$ be the approximate roots common to α and β. Let $\mathbf{Q}^{\gamma_1}, \mathbf{Q}^{\gamma_2}, \dots$ be the list of monomials in $\mathbf{Q}_\Lambda$, arranged in the increasing order of the ν_α values. There exists an ordinal s such that $< \alpha, \beta >$ is generated by the set $\{\mathbf{Q}^{\gamma_j};\ j \leq s,\ \mathbf{Q}^{\gamma_j} \in < \alpha, \beta >\}$. Let σ be the unique ordinal such that $\mathbf{Q}^{\gamma_a} \notin < \alpha, \beta >$ for $a < \sigma$ and $\mathbf{Q}^{\gamma_\sigma}, \mathbf{Q}^{\gamma_{\sigma+1}}, \dots \in < \alpha, \beta >$.

Next, we study the standard form of f_i of level $\nu_\alpha(< \alpha, \beta >)$. In the case when A is complete, this standard form may contain infinitely many generalized monomials $\mathbf{Q}^\gamma$. Since A is noetherian, we can choose a finite subset $\mathbf{Q}^{\epsilon_{ji}}$, $1 \leq j \leq n_i$, of these monomials such that all of the others lie in the ideal $(\mathbf{Q}^{\epsilon_{ji}}, 1 \leq j \leq n_i)A$. For $i \in \{1, \dots, r\}$, let

$$f_i = \sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}} + \sum_{j'=1}^{n_i} c_{j'i} \mathbf{Q}^{\epsilon_{j'i}} \tag{75}$$

be the standard expansion of f_i of level $\nu_\alpha(< \alpha, \beta >)$ where $\nu_\alpha(\mathbf{Q}^{\theta_{ji}}) = \nu_\alpha(f_i) < \nu_\alpha(\mathbf{Q}^{\epsilon_{j'i}})$ for all $j \in \{1, \dots, m_i\}$ and $j' \in \{1, \dots, n_i\}$.

Remark 2.2.1 *1. If $k = k_\alpha$ (in particular, if k is real closed), then $m_i = 1$.*
2. By Proposition 1.5.5, $\sum_{j=1}^{m_i} b_{ji} in_{\nu_\alpha} \mathbf{Q}^{\theta_{ji}} \neq 0$.

Conjecture 2.2.2 *1. Let*

$$C = \left\{ \delta \in SperA \;\middle|\; \begin{array}{ll} \nu_\delta(\mathbf{Q}^{\theta_{ji}}) < \nu_\delta(\mathbf{Q}^{\epsilon_{j'i}}) & \text{for all } j \in \{1, \dots, m_i\},\ j' \in \{1, \dots, n_i\} \\ sgn_\delta(Q_q) = sgn_\alpha(Q_q) & \text{for all } Q_q \text{ appearing in } \mathbf{Q}^{\theta_{ji}} \\ sgn_\delta(\sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}}) = & sgn_\alpha(\sum_{j=1}^{m_i} b_{ji} \mathbf{Q}^{\theta_{ji}}) \end{array} \right\}. \tag{76}$$

Then C is connected.

2. Let C' defined by the inequalities

$$\left|\sum_{j=1}^{m_i} b_{ji}\mathbf{Q}^{\theta_{ji}}\right| >_\delta n_i|\mathbf{Q}^{\epsilon_{j'i}}| \ \forall i \in \{1,\dots,r\},\ \forall j' \in \{1,\dots,n_i\} \tag{77}$$

and the two sign conditions appearing in (76). Then C' is connected.

Remark 2.2.3 *1. We have $\alpha, \beta \in C$.*

2. $C \cap \{f_1 \cdots f_r = 0\} = \emptyset$. Indeed, inequalities (76) imply that, for every $\delta \in C$, f_i has the same sign as $\sum_{j=1}^{m_i} b_{ji}\mathbf{Q}^{\theta_{ji}}$; in particular, none of the f_i vanish on C.

3. Either of those conjectures implies the Connectedness Conjecture.

Part 3. A proof of the conjecture for arbitrary regular 2-dimensional rings.

We start with a general plan of the proof and an outline of different sections of Part 3. In §3.1 we recall Zariski's theory of complete ideals. We explain how the construction of approximate roots in arbitrary dimension restricts to the special case of dimension 2 (and that the standard construction in dimension 2 is, indeed, recovered from the general one as a special case) and prove some general lemmas about approximate roots in regular two dimensional local rings and their behaviour under sequences of point blowings up. In §3.2 we define the notion of real geometric surfaces which are glued from affine charts of the form Sper A_j, where A_j is a regular two-dimensional ring, in order to be able to talk about point blowings up of Sper A. We also define the notion of a segment on the exceptional divisor of a blowing up and prove that such a segment is connected; another notion useful later in the proof is that of a maximal segment. One slightly delicate point here is that since the residue field k of A is not assumed real closed we need to fix an order on k and always restrict attention to points of the real spectra of various A_j which induce the given order on k. The bulk of the proof *per se* is contained in §§3.3–3.5. As explained above, our problem is one of proving connectedness (resp. definable connectedness) of the set C.

In §3.3 we use Zariski's theory and other results from §3.1 to construct a sequence of point blowings up which transform C into a quadrant, that is, a set $\tilde{U}$ of all points δ of a suitable affine chart Sper A_j centered at the origin satisfying either $x'(\delta) > 0$, $y'(\delta) > 0$, or just $x'(\delta) > 0$. In §3.4 we use results from [3] to prove connectedness of $\tilde{U}$ by reducing it to that of a quadrant in the usual Euclidean space, assuming that A is excellent. In §3.5 we prove the definable connectedness of $\tilde{U}$ (without any excellence assumptions) after introducing a new object called the graph associated to $\tilde{U}$ and a finite sequence of point blowings up of Sper A.

3.1 Approximate roots in dimension 2 and Zariski's theory.

In the special case of regular 2-dimensional local rings, the theory of approximate roots is well known: see, for instance [45], Appendix 5 or [36]. We briefly recall the construction here since it is much simpler than in the general case.

We start with two purely combinatorial lemmas about semigroups. Take an integer $g \geq 2$.

Lemma 3.1.1 *Let $\beta_1, \beta_2 \dots, \beta_g$ be positive elements in some ordered group. Let α_j, $j \in \{2, \dots, g\}$ be positive integers. Assume*

$$\beta_i \geq \alpha_{i-1}\beta_{i-1},\ i \in \{3, \dots g\}. \tag{78}$$

Let $\gamma_1, \dots, \gamma_g$ be integers such that $0 \leq \gamma_j < \alpha_j$ for $2 \leq j \leq g$ and $\sum_{j=1}^{g} \gamma_j\beta_j \geq \alpha_g\beta_g$. Then $\gamma_1 > 0$.

Proof : We prove by descending induction that $\sum_{j=1}^{i}\gamma_j\beta_j \geq \alpha_i\beta_i$ for $i \geq 2$. The case $i = g$ is given by hypothesis. Assume then that $\sum_{j=1}^{i+1}\gamma_j\beta_j \geq \alpha_{i+1}\beta_{i+1}$. Subtracting $\gamma_{i+1}\beta_{i+1}$ and using the fact that $\gamma_{i+1} < \alpha_{i+1}$, we obtain $\sum_{j=1}^{i}\gamma_j\beta_j \geq (\alpha_{i+1}-\gamma_{i+1})\beta_{i+1} \geq \alpha_i\beta_i$. This completes the induction. So for $i = 2$, we obtain $\gamma_1\beta_1 + \gamma_2\beta_2 \geq \alpha_2\beta_2$; subtracting $\gamma_2\beta_2$ and using the fact that $\gamma_2 < \alpha_2$, we get $\gamma_1\beta_1 \geq (\alpha_2 - \gamma_2)\beta_2 > 0$, hence $\gamma_1 > 0$. □

Notation. Let $\beta_1, \beta_2 \ldots, \beta_g$ be positive elements in some ordered group. We will denote by $(\beta_1, \ldots, \beta_{i-1})$ the group generated by $\beta_1, \ldots, \beta_{i-1}$ and by $sg(\beta_1, \ldots, \beta_{i-1})$ the semigroup generated by $\beta_1, \ldots, \beta_{i-1}$, that is, the semigroup formed by all the $\mathbb{N}$-linear combinations of $\beta_1, \ldots, \beta_{i-1}$. For $i \in \{2, \ldots, g\}$, α'_i will denote the smallest positive integer such that $\alpha'_i\beta_i \in (\beta_1, \ldots, \beta_{i-1})$. If there is no such integer, we put $\alpha'_i = \infty$. Write

$$\alpha'_i\beta_i = \sum_{j=1}^{i-1}\alpha_{ji}\beta_j \quad \text{where } \alpha_{ji} \in \mathbb{Z}. \tag{79}$$

Lemma 3.1.2 *Let $\beta_1, \beta_2 \ldots, \beta_g$ be positive rational numbers such that $\beta_g \geq \alpha'_{g-1}\beta_{g-1}$. If $g \geq 3$, assume that*

$$\{a \in (\beta_1, \ldots, \beta_{g-1}) \mid a \geq \alpha'_{g-1}\beta_{g-1}\} = \{a \in sg(\beta_1, \ldots, \beta_{g-1}) \mid a \geq \alpha'_{g-1}\beta_{g-1}\}; \tag{80}$$

in particular, we can choose $\alpha_{jg} \geq 0$ for all $j \in \{1, \ldots, g-1\}$ in (79) when $i = g$. Then

$$\{a \in (\beta_1, \ldots, \beta_g) \mid a \geq \alpha'_g\beta_g\} = \{a \in sg(\beta_1, \ldots, \beta_g) \mid a \geq \alpha'_g\beta_g\}. \tag{81}$$

Proof. Multiplying all the β_i by the same rational number does not change the problem, so we may assume that $\beta_1, \beta_2, \ldots, \beta_g$ are positive integers, such that $\gcd(\beta_1, \beta_2, \ldots, \beta_g) = 1$.

For $g = 2$, we have $\alpha'_2 = \beta_1$. If $a \in (\beta_1)$ and $a \geq \beta_1\beta_2$, then $a > 0$, hence $a \in sg(\beta_1)$; thus

$$\{a \in (\beta_1) \mid a \geq \beta_1\beta_2\} \subset \{a \in sg(\beta_1) \mid a \geq \beta_1\beta_2\},$$

the opposite inclusion being obvious.

Assume that $g \geq 3$. Write

$$\alpha'_g\beta_g = \sum_{j=1}^{g-1}\alpha_{jg}\beta_j. \tag{82}$$

To prove (81), let $\beta = \gamma_1\beta_1 + \gamma_2\beta_2 + \cdots + \gamma_g\beta_g$ be an element of

$$\{a \in (\beta_1, \ldots \beta_g) \mid a \geq \alpha'_g\beta_g\}.$$

Using the relation (82) we can write, for each $n \in \mathbb{Z}$,

$$\beta = \sum_{j=1}^{g-1}(\gamma_j - n\alpha_{jg})\beta_j + (\gamma_g + n\alpha'_g)\beta_g = \sum_{j=1}^{g-1}\gamma'_j\beta_j + (\gamma_g + n\alpha'_g)\beta_g.$$

After replacing γ_g by $\gamma_g + n\alpha'_g$ for a suitable $n \in \mathbb{Z}$, we may assume that $0 \leq \gamma_g < \alpha'_g$. Since $\beta \geq \alpha'_g\beta_g$, this implies that

$$\sum_{j=1}^{g-1}\gamma_j\beta_j \geq \beta_g \geq \alpha'_{g-1}\beta_{g-1}. \tag{83}$$

By (80) , we may take $\gamma_j \geq 0$ in (83). This completes the proof of the Lemma. □

Corollary 3.1.3 *Let $\beta_1, \ldots, \beta_g$ be positive rational numbers satisfying*

$$\beta_i \geq \alpha'_{i-1}\beta_{i-1}, \ i \in \{3, \ldots g\}. \tag{84}$$

Then equalities (80) and (81) hold.

Proof : For $i = 3$, (80) is immediate. Now the corollary follows from Lemma 3.1.2 by induction on i. $\square$

Let ν be a valuation centered at A and let (x, y) be a ν-prepared system of coordinates, such that $\nu(x) = \nu(\mathfrak{m})$. In what follows, we will omit the description of $\mathcal{V}(\gamma), \Lambda(\gamma), \Theta(\gamma)$, since in the simplified situation of $n = 2$, the sets $\Psi(\gamma)$ suffice to carry out the entire construction.

Put $Q_1 = x$, $Ex(Q_1) = x$, $Q_2 = y$, $Ex(Q_2) = y$ and $\beta_i = \nu(Q_i)$, $i \in \{1, 2\}$. If β_1, β_2 are rationally independent, then $\alpha'_2 = \infty$ and the construction stops, there are no more approximate roots. In this case, all the ν-ideals are generated by monomials in (x, y). Assume then $\alpha'_2 < \infty$. This means that there is a relation $\alpha'_2\beta_2 = \alpha_{12}\beta_1$ for a positive integer α_{12}.

Let α'_2 and α_{12} be as above. Let $\Psi(\beta_1) = \emptyset$. For $\gamma \in \Phi$, $\beta_1 < \gamma < \beta_2$, $\Psi(\gamma) = \{x\}$ and $\Psi(\beta_2+) = \{x, y\}$. Let $k_1 = k$, $k_2 = k\left(\dfrac{\mathrm{in}_\nu\left(Q_2^{\alpha'_2}\right)}{\mathrm{in}_\nu\left(Q_1^{\alpha_{12}}\right)}\right)$.

We prove that (80) is satisfied for $i = 3$. Let $\beta = \gamma_1\beta_1 + \gamma_2\beta_2$ be an element of

$$\{a \in (\beta_1, \beta_2) \mid a \geq \alpha'_2\beta_2\}.$$

As $\alpha'_2\beta_2 = \alpha_{12}\beta_1$, we have, for each $n \in \mathbb{Z}$, $\beta = (\gamma_1 - n\alpha_{12})\beta_1 + (\gamma_2 + n\alpha'_2)\beta_2$. After replacing γ_2 by $\gamma_2 + n\alpha'_2$ for a suitable n, we may assume that $0 \leq \gamma_2 < \alpha'_2$. Since $\beta \geq \alpha'_2\beta_2$, this implies that $\gamma_1 \geq 0$.

Then we have $\nu\left(Q_2^{\alpha'_2}\right) = \nu\left(Q_1^{\alpha_{12}}\right)$, hence the image of $\dfrac{\mathrm{in}_\nu(Q_2^{\alpha'_2})}{\mathrm{in}_\nu(Q_1^{\alpha_{12}})}$ in k_ν is not zero. If $\dfrac{\mathrm{in}_\nu(Q_2^{\alpha'_2})}{\mathrm{in}_\nu(Q_1^{\alpha_{12}})}$ is algebraic over k, this means that it satisfies an algebraic equation of the form

$$X^d + \overline{a}_1 X^{d-1} + \cdots + \overline{a}_d = 0,\ \overline{a}_i \in k. \tag{85}$$

Let a_i be a representative of $\overline{a}_i$ in A. Let $\alpha_2 = d\alpha'_2$ and

$$Q_3^{(1)} = Q_2^{\alpha_2} + \sum_{\ell=1}^{d} a_\ell Q_2^{\alpha'_2(d-\ell)} Q_1^{\alpha_{12}\ell}. \tag{86}$$

The expression $Ex\left(Q_3^{(1)}\right)$ is just the right hand side of this formula.

Let $\beta_3^{(1)} = \nu\left(Q_3^{(1)}\right)$, then $\beta_3^{(1)} > \nu(Q_2^{\alpha_2}) = \alpha_2\beta_2 \geq \alpha'_2\beta_2$ and the elements $\left(\beta_1, \beta_2, \beta_3^{(1)}\right)$ satisfy the conclusion of Lemma 3.1.2.

By construction, $\alpha_2\beta_2$ is the smallest element of Φ such that the monomials

$$\{Q_1^{\gamma_1}Q_2^{\gamma_2} \mid \nu(Q_1^{\gamma_1}Q_2^{\gamma_2}) = \gamma_1\beta_1 + \gamma_2\beta_2 = \alpha_2\beta_2\}$$

are k-linearly dependent. The unique k-linear dependence relation is given by $Q_3^{(1)}$. Hence, according to the general construction of §2, we have $\Theta(\beta) = \{Q_3^{(1)}\}$ for $\alpha_2\beta_2 \geq \beta \geq \beta_3^{(1)}$ and $\Psi(\beta_3^{(1)}+) = \{Q_1, Q_2, Q_3^{(1)}\}$.

Assume that $i \geq 3$ and that elements $Q_1, \ldots, Q_{i-1}, Q_i^{(j)}$ are already defined. Let

$$\beta_q = \nu(Q_q), \tag{87}$$

$$\beta_i^{(j)} = \nu\left(Q_i^{(j)}\right). \tag{88}$$

Assume that the initial form $\mathrm{in}_\nu Q_q$ is algebraic over $k[\mathrm{in}_\nu Q_1, \ldots, \mathrm{in}_\nu Q_{q-1}]$ for $q \in \{2, \ldots, i-1\}$. Let α_q denote the degree of its minimal polynomial. Note that, in particular, all of

$\beta_2,\dots,\beta_{i-1}$ are rational multiples of β_1. Assume that $\beta_q > \alpha_{q-1}\beta_{q-1}$, $q\in\{3,\dots,i-1\}$ and $\beta_i^{(j)} > \alpha_{i-1}\beta_{i-1}$. Assume that, in the notation of §1.2, we have

$$\Psi\left(\beta_{i\ +}^{(j)}\right) = \left\{Q_1,\dots,Q_{i-1},Q_i^{(j)}\right\}.$$

A monomial $\prod\limits_{\ell=1}^{i-1} Q_\ell^{\epsilon_\ell}$ is **standard** if

$$0\le\epsilon_\ell<\alpha_\ell \text{ for } \ell\in\{2,\dots,i-1\}. \tag{89}$$

This allows us to extend the notion of standard to monomials with $\epsilon_1<0$: such a monomial is called standard if (89) is satisfied. Similarly, we may talk about standard monomials in $\mathrm{in}_\nu Q_1,\dots,\mathrm{in}_\nu Q_{i-1}$.

Assume, in addition, that we have defined elements $z_2,\dots,z_{i-1}\in k_\nu$, algebraic over k, where z_ℓ is a k-linear combination of standard monomials in $\mathrm{in}_\nu Q_1,\dots,\mathrm{in}_\nu Q_\ell$ of degree 0. Let $k_\ell = k(z_2,\dots,z_\ell)$. We obtain a tower of finite field extensions $k=k_1\subset k_2\dots\subset k_{i-1}\subset k_\nu$.

If $\mathrm{in}_\nu Q_i^{(j)}$ is transcendental over $k[\mathrm{in}_\nu Q_1,\dots,\mathrm{in}_\nu Q_{i-1}]$, put $Q_i=Q_i^{(j)}$ and the construction stops.

Assume $\mathrm{in}_\nu Q_i^{(j)}$ is algebraic over $k[\mathrm{in}_\nu Q_1,\dots,\mathrm{in}_\nu Q_{i-1}]$. Then $\beta_i^{(j)}\in\sum\limits_{q=1}^{i-1}\mathbb{Q}\beta_q$. Let $\alpha_i^{;(j)}$ be the smallest positive integer such that $\alpha_i^{;(j)}\beta_i^{(j)}\in(\beta_1,\dots,\beta_{i-1})$.

Then $\nu\left(\left(Q_i^{(j)}\right)^{\alpha_i^{;(j)}}\right)=\nu\left(\prod\limits_{r=1}^{i-1}Q_r^{\alpha_{ri}^{(j)}}\right)$, hence the image of $\dfrac{\mathrm{in}_\nu\left(Q_i^{(j)}\right)^{\alpha_i^{;(j)}}}{\mathrm{in}_\nu\prod_{r=1}^{i-1}Q_r^{\alpha_{ri}^{(j)}}}$ in k_ν is not zero.

By Corollary 3.1.3, we may take $\alpha_{ri}^{(j)}\ge0$ for $1\le r\le i-1$.

The assumption on $\mathrm{in}_\nu Q_i^{(j)}$ implies that $\dfrac{\mathrm{in}_\nu\left(Q_i^{(j)}\right)^{\alpha_i^{;(j)}}}{\mathrm{in}_\nu\prod_{r=1}^{i-1}Q_r^{\alpha_{ri}^{(j)}}}$ satisfies an algebraic equation of the form

$$X^d+\overline{a}_1X^{d-1}+\dots+\overline{a}_d=0,\ \overline{a}_\ell\in k_{i-1}. \tag{90}$$

For $\ell\in\{1,\dots,d\}$, write

$$\overline{a}_\ell\left(\prod_{r=1}^{i-1}\mathrm{in}_\nu Q_r^{\alpha_{ri}^{(j)}}\right)^\ell=\sum_{\gamma=(\gamma_1,\dots,\gamma_{i-1})}\overline{b}_{\ell\gamma}\prod_{r=1}^{i-1}\mathrm{in}_\nu Q_r^{\gamma_r} \tag{91}$$

as a k-linear combination of standard monomials. By Lemma 3.1.1, we have $\gamma_1\ge0$ whenever $\overline{b}_{\ell\gamma}\ne0$.

Let $b_{\ell\gamma}$ be a representative of $\overline{b}_{\ell\gamma}$ in A. Let $\alpha_i^{(j)}=d\alpha_i^{;(j)}$ and

$$Q=\left(Q_i^{(j)}\right)^{\alpha_i^{(j)}}+\sum_{\ell=1}^d\left(\sum_{\gamma=(\gamma_1,\dots,\gamma_{i-1})}b_{\ell\gamma}\prod_{r=1}^{i-1}Q_r^{\gamma_r}\right)Q_i^{\alpha_i^{;(j)(d-\ell)}}. \tag{92}$$

Then

$$\nu(Q)>\nu\left(\left(Q_i^{(j)}\right)^{\alpha_i^{(j)}}\right)=\alpha_i^{(j)}\beta_i^{(j)}\ge\alpha_i^{;(j)}\beta_i^{(j)}>\alpha_{i-1}\beta_{i-1}\ge\alpha_{i-1}^{;}\beta_{i-1}. \tag{93}$$

If $\mathrm{in}_\nu Q_i^{(j)}\notin k[\mathrm{in}_\nu Q_1,\dots,\mathrm{in}_\nu Q_{i-1}]$ (which is equivalent to saying that $\alpha_i^{(j)}>1$), put $Q_i=Q_i^{(j)}$, $Ex(Q_i)=Ex(Q_i^{(j)})$, $\beta_i=\beta_i^{(j)}$, $\alpha_i=\alpha_i^{(j)}$, $\alpha_i^;=\alpha_i^{;(j)}$, $Q_{i+1}^{(1)}=Q$, $\beta_{i+1}^{(1)}=\nu\left(Q_{i+1}^{(1)}\right)$.

Formulae (92) and (93) become

$$Q_{i+1}^{(1)} = Q_i^{\alpha_i} + \sum_{\ell=1}^{d} \left(\sum_{\gamma=(\gamma_1,\dots,\gamma_{i-1})} b_{\ell\gamma} \prod_{r=1}^{i-1} Q_r^{\gamma_r} \right) Q_i^{\alpha_i^{\prime}(d-\ell)}. \tag{94}$$

and

$$\beta_{i+1}^{(1)} = \nu\left(Q_{i+1}^{(1)}\right) > \nu(Q_i^{\alpha_i}) = \alpha_i\beta_i \geq \alpha_i^{\prime}\beta_i. \tag{95}$$

The expression $Ex\left(Q_{i+1}^{(1)}\right)$ is just the right hand side of (94).

For $\beta_i^{(j)} < \gamma \leq \beta_{i+1}^{(1)}$, we have $\Psi(\gamma) = \Psi\left(\beta_i^{(j)}+\right)$ and $\Psi\left(\beta_{i+1}^{(1)}+\right) = \left\{Q_1, \dots, Q_i, Q_{i+1}^{(1)}\right\}$. Moreover, the elements $\left(\beta_1, \dots, \beta_{i+1}^{(1)}\right)$ satisfy the hypothesis of Corollary 3.1.3, hence also its conclusion.
If $\mathrm{in}_\nu Q_i^{(j)} \in k[\mathrm{in}_\nu Q_1, \dots, \mathrm{in}_\nu Q_{i-1}]$ (which is equivalent to saying that $\alpha_i^{(j)} = 1$), put $Q_i^{(j+1)} = Q$ and $\beta_i^{(j+1)} = \nu\left(Q_i^{(j+1)}\right)$.
Formulae (92) and (93) become

$$Q_i^{(j+1)} = Q_i^{(j)} + \sum_{\gamma=(\gamma_1,\dots,\gamma_{i-1})} b_{1\gamma} \prod_{r=1}^{i-1} Q_r^{\gamma_r} \tag{96}$$

and

$$\beta_i^{(j+1)} = \nu\left(Q_i^{(j+1)}\right) > \beta_i^{(j)} > \alpha_{i-1}^{\prime}\beta_{i-1}. \tag{97}$$

The expression $Ex\left(Q_i^{(j+1)}\right)$ is just the right hand side of (96).

For $\beta_i^{(j)} < \gamma \leq \beta_i^{(j+1)}$ we have

$$\Psi(\gamma) = \Psi\left(\beta_i^j+\right) \quad \text{and} \tag{98}$$

$$\Psi\left(\beta_i^{(j+1)}+\right) = \left\{Q_1, \dots, Q_{i-1}, Q_i^{(j+1)}\right\}. \tag{99}$$

Moreover, the elements $\left(\beta_1, \dots, \beta_{i-1}, \beta_i^{(j+1)}\right)$ satisfy the hypothesis of Corollary 3.1.3, hence also its conclusion.

Remark 3.1.4 *Either the process stops after a finite number of steps or we obtain an infinite sequence*

$$\mathbf{Q} = Q_1, Q_2, \dots, Q_i, \dots \tag{100}$$

or a sequence

$$\mathbf{Q} = Q_1, Q_2, \dots, Q_{i-1}, Q_i^{(j)}, \qquad j \in \mathbb{N}. \tag{101}$$

In the case when $\mathbf{Q}$ is given by (100), it is a system of approximate roots, whether or not A is complete. In the case (101) assume, in addition, that the ring A is $\mathfrak{m}$-adically complete. In that case,

$$Q_\infty = \lim_{j\to\infty} Q_i^{(j)}$$

is a well defined element of A and $\mathbf{Q} \cup \{Q_\infty\}$ is a system of approximate roots.

We recall some basic facts from Zariski's theory of complete ideals in regular two-dimensional local rings.

Let $(A, \mathfrak{m})$ be a regular 2-dimensional local ring, x, y a regular system of parameters and let ν be a valuation centered at A.

Definition 3.1.5 *An ideal $\mathcal{I}$ in a normal ring B is said to be integrally closed or complete if it contains all the elements z of B satisfying a monic equation of the form*

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$$

where $a_{n-i} \in \mathcal{I}^{n-i}$.

An ideal $\mathcal{I}$ in A is said to be simple if it cannot be factored in a non trivial way as a product of two other ideals.

A ***local blowing up of*** *A* ***with respect to*** *ν* ***along*** *$\mathfrak{m}$ is the map $A \to A[\frac{y}{x}]_{\mathfrak{m}_1}$, where $\mathfrak{m}_1$ is the center of ν in $A[\frac{y}{x}]$.*

For an element $f \in A$, we have $x^{\nu_\mathfrak{m}(f)} \mid f$ in $A[\frac{y}{x}]_\mathfrak{m}$. The ***strict transform*** *of f in $A[\frac{y}{x}]_\mathfrak{m}$ is the element $x^{-\nu_\mathfrak{m}(f)}f$.*

Remark 3.1.6 *Any ν-ideal is a complete ideal.*

Now let I be a simple $\mathfrak{m}$-primary ν-ideal. Then
(1) The set

$$\mathfrak{m} = I_0 \supset I_1 \supset \cdots \supset I_\ell = I$$

of simple ν-ideals of A containing I is entirely determined by I (it does not depend on ν).
(2) Let $A \to A_1$ be the local blowing up with respect to ν along $\mathfrak{m}$ and, for $i \geq 1$, let I'_i be the transform of I_i (that is, $I'_i = x^{-\mu} I_i A_1$ with $\mu = ord_\mathfrak{m} I_i$). Then

$$\mathfrak{m}_1 = I'_1 \supset I'_2 \supset \cdots \supset I'_{\ell-1} = I'$$

is the set .
(3) Iterating this procedure ℓ-times, we obtain a sequence of local blowing ups

$$(A, \mathfrak{m}) \to (A_1, \mathfrak{m}_1) \to \cdots \to (A_\ell, \mathfrak{m}_\ell) \tag{102}$$

such that the transform $I^{(\ell)}$ of I is $\mathfrak{m}_\ell$. For any $f \in A \setminus I$, the strict transform of f in A_ℓ is a unit of A_ℓ.

We recall the following general result from the theory of approximate roots in regular 2-dimensional local rings ([36]).

Let A be a 2-dimensional regular local ring, ν a valuation on A. Now let Q_k, $k = 1, \dots, g+1$ be the approximate roots of ν such that $Q_1, \dots, Q_g \notin I$ and $Q_{g+1} \in I$. Each I_i is generated by the generalized monomials $\prod Q_j^{\gamma_j}$, $\gamma_j \in \mathbb{N}$, such that $\sum \gamma_j \beta_j \geq \nu(I_i)$.

Proposition 3.1.7 *There exist natural numbers $\ell_1 < \ell_2 < \cdots < \ell_g \leq \ell$ and a regular system of parameters x_{ℓ_i}, y_{ℓ_i} for each $i \in \{1, \dots, g\}$ having the following properties :*

(1) x_{ℓ_i} is a monomial of the form $\prod\limits_{j=1}^{i-1} Q_j^{\gamma_j}$, $\gamma_j \in \mathbb{N}$,
(2) y_{ℓ_i} is the strict transform of Q_i in A_{ℓ_i},
(3) $Q_1, \dots, Q_{i-1}$ are monomials in x_{ℓ_i}, y_{ℓ_i} times a unit of $(A_{\ell_i})_{(x_{\ell_i}, y_{\ell_i})}$.

For $\alpha, \beta \in \mathrm{Sper}A$, let $Q_1, \dots, Q_s$ be the approximate roots common to α and β.

Corollary 3.1.8 *If $i \leq s$, both ν_α and ν_β are centered at (x_{ℓ_i}, y_{ℓ_i}).*

Let A be a 2-dimensional regular local ring, ν a valuation on A. Keep all the above notations.
Convention : below, we adopt the convention that $\alpha_1 = 1$.

Lemma 3.1.9 *For $i \geq 3$, $\nu_\mathfrak{m}(Q_i) = \prod_{j=1}^{i-1} \alpha_j$.*

Proof : Let $i = 3$, then we can write $Q_3 = y^{\alpha_2} + \sum c_{rs} x^r y^s$ where c_{rs} is a unit in A, with $\nu(x^r y^s) \geq \alpha_2 \nu(y)$. As $\nu(y) \geq \nu(x)$, this implies $\nu_{\mathfrak{m}}(Q_3) = \alpha_2$.

Recall (cf. (92)) that

$$Q_{i+1} = Q_i^{\alpha_i} + \sum_{\ell=1}^{d} \left(\sum_{\gamma=(\gamma_1,\dots,\gamma_{i-1})} b_{\ell\gamma} \prod_{r=1}^{i-1} Q_r^{\gamma_r} \right) Q_i^{\alpha_i'(d-\ell)}.$$

Now to prove the lemma, it suffices to prove that

$$\alpha_i \nu_{\mathfrak{m}}(Q_i) \leq \nu_{\mathfrak{m}} \left(\left(\prod_{r=1}^{i-1} Q_r^{\gamma_r} \right) Q_i^{\alpha_i'(d-\ell)} \right) \tag{103}$$

for all ℓ and γ such that $b_{\ell\gamma} \neq 0$.
First remark that, according to the inequalities (95) and (97), we deduce by an easy induction on $i - \ell$ that

$$\frac{\beta_i}{\prod_{q=\ell}^{i-1} \alpha_q} \geq \beta_\ell. \tag{104}$$

We have $\alpha_i \beta_i = \sum_{j=1}^{i-1} \gamma_j \beta_j + \alpha_i'(d-\ell)\beta_i$, so

$$\alpha_i' \ell \beta_i = \sum_{j=1}^{i-1} \gamma_j \beta_j \leq \sum_{j=1}^{i-1} \gamma_j \frac{\beta_i}{\prod_{q=j}^{i-1} \alpha_q}$$

by (104).

Dividing both sides by $\frac{\beta_i}{\prod_{q=1}^{i-1} \alpha_q}$, we get

$$\alpha_i' \ell \prod_{q=1}^{i-1} \alpha_q \leq \sum_{j=1}^{i-1} \gamma_j \prod_{q=1}^{j-1} \alpha_q. \tag{105}$$

By the induction assumption, the left hand side equals $\nu_{\mathfrak{m}}(Q_i^{\alpha_i' \ell})$ and the right hand side equals $\nu_{\mathfrak{m}}(\prod_{j=1}^{i-1} Q_j^{\gamma_j})$. Therefore inequality (103) follows from inequality (105).$\Box$

In what follows, we study standard monomials $\prod_{j=1}^{i} Q_j^{\gamma_j}$, with $i < s$, that is monomials such that $0 \leq \gamma_j < \alpha_j$ for $j \in \{2, \dots, i\}$.

Corollary 3.1.10 *Consider two standard monomials $\prod_{j=1}^{i} Q_j^{\gamma_j}$ and $\prod_{j=1}^{i} Q_j^{\gamma_j'}$ such that $(\gamma_i, \gamma_{i-1}, \dots, \gamma_1) <_{lex} (\gamma_i', \gamma_{i-1}', \dots, \gamma_1')$ and having the same ν-value. We have*

$$\nu_{\mathfrak{m}} \left(\prod_{j=1}^{i} Q_j^{\gamma_j} \right) > \nu_{\mathfrak{m}} \left(\prod_{j=1}^{i} Q_j^{\gamma_j'} \right).$$

Let $n = \nu_{\mathfrak{m}}(Q_3)$; note that $\alpha_2 = n$. Moreover $[k_2 : k] \mid n$ and $[k_2 : k] = n$ if and only if $\beta_1 \mid \beta_2$.

Corollary 3.1.11 *Consider two standard monomials $\prod_{j=1}^{i} Q_j^{\gamma_j}$ and $\prod_{j=1}^{i} Q_j^{\gamma_j'}$, with $3 \leq i < s$, such that $(\gamma_i, \gamma_{i-1}, \dots, \gamma_3) <_{lex} (\gamma_i', \gamma_{i-1}', \dots, \gamma_3')$. We have*

$$\nu_{\mathfrak{m}} \left(\prod_{j=3}^{i} Q_j^{\gamma_j} \right) \leq \nu_{\mathfrak{m}} \left(\prod_{j=3}^{i} Q_j^{\gamma_j'} \right) - n.$$

Proof : Let $j \geq 3$ be the greatest integer such that $\gamma_j < \gamma'_j$. We have

$$\begin{aligned}\nu_{\mathfrak{m}}\left(\prod_{j=3}^{i} Q_j^{\gamma'_j}\right) - \nu_{\mathfrak{m}}\left(\prod_{j=3}^{i} Q_j^{\gamma_j}\right) &= \sum_{\ell=3}^{j} \gamma'_\ell \prod_{q=1}^{\ell-1} \alpha_q - \sum_{\ell=3}^{j} \gamma_\ell \prod_{q=1}^{\ell-1} \alpha_q \\ &= (\gamma'_j - \gamma_j) \prod_{q=1}^{j-1} \alpha_q + \sum_{\ell=3}^{j-1} (\gamma'_\ell - \gamma_\ell) \prod_{q=1}^{\ell-1} \alpha_q.\end{aligned}$$

Claim: For $j \geq 4$ and $c_\ell < \alpha_\ell$, we have

$$\sum_{\ell=3}^{j-1} c_\ell \prod_{q=1}^{\ell-1} \alpha_q < \prod_{q=1}^{j-1} \alpha_q. \tag{106}$$

Proof of Claim: By induction on j. For $j = 4$, the inequality is immediate. Assume the Claim is true for $j-1$. The left hand side of (106) can be rewritten as

$$\sum_{\ell=3}^{j-1} c_\ell \prod_{q=1}^{\ell-1} \alpha_q = \sum_{\ell=3}^{j-2} c_\ell \prod_{q=1}^{\ell-1} \alpha_q + c_{j-1} \prod_{q=1}^{j-2} \alpha_q < \prod_{q=1}^{j-2} \alpha_q + c_{j-1} \prod_{q=1}^{j-2} \alpha_q \leq \prod_{q=1}^{j-1} \alpha_q.$$

The Claim is proved.

The monomials being standard, $0 \leq \gamma_\ell, \gamma'_\ell < \alpha_\ell$, so $\gamma'_\ell - \gamma_\ell > -\alpha_\ell$ and applying the Claim, we deduce that

$$\sum_{\ell=3}^{j-1} (\gamma'_\ell - \gamma_\ell) \prod_{q=1}^{\ell-1} \alpha_q > -\prod_{q=1}^{j-1} \alpha_q.$$

Since $\gamma'_j - \gamma_j \geq 1$, we get

$$(\gamma'_j - \gamma_j) \prod_{q=1}^{j-1} \alpha_q + \sum_{\ell=3}^{j-1} (\gamma'_\ell - \gamma_\ell) \prod_{q=1}^{\ell-1} \alpha_q > 0.$$

Each term being an integer divisible by α_2, the above expression is greater or equal to $\alpha_2 = n$. $\square$

3.2 Real geometric surfaces and their blowings up

Let A be a ring and U an open subset of $\mathrm{Sper}(A)$. Let S_U denote the multiplicative set

$$S_U = \{g \in A \mid g(\alpha) \neq 0 \text{ for all } \alpha \in U\}.$$

Let $A_U = A_{S_U}$. We have a natural ring homomorphism

$$\rho_U : A_U \to \prod_{\alpha \in U} A(\alpha).$$

Define the ring O_U to be the ring of all maps

$$f : U \to \coprod_{\alpha \in U} A(\alpha)$$

satisfying the following conditions :

(1) $\forall \alpha \in U$, $f(\alpha) \in A(\alpha)$;

(2) there exists an open covering

$$U = \bigcup_{i \in \Lambda} U_i \tag{107}$$

and, for each i, an element $f_i \in A_{U_i}$ such that $\forall \beta \in U_i$, we have $\rho_{U_i}(f_i)_\beta = f(\beta)$.

The functor which sends U to O_U makes $\mathrm{Sper}(A)$ into a locally ringed space which we will call an *affine real geometric space*. This notion is inspired by the notion of real closed spaces defined by Niels Schwartz ([35]).

From now till the end of this section we will assume that all our rings are integral domains.

Remark 3.2.1 *Note that $\iota : A_U \hookrightarrow O_U$ and, if U is connected, this inclusion becomes an equality. Indeed, consider an element $f \in O_U$, the open covering (107) and the local representatives $f_i \in A_{U_i}$ of f as above. Let K denote the common field of fractions of A and all of the A_U. Finding an element $g \in A_U$ such that $\iota(g) = f$ amounts to proving that for each $i, j \in \Lambda$ we have*

$$f_i = f_j, \tag{108}$$

viewed as elements of K. By connectedness of U, it is sufficient to prove (108) under the assumption that $U_i \cap U_j \neq \emptyset$. Take a non-empty basic open subset $V \subset U_i \cap U_j$, defined by finitely many inequalities $V = \{\alpha \in Sper\ A \mid g_1(\alpha) > 0, \dots, g_s(\alpha) > 0\}$. Since $V \neq \emptyset$, Propositions 4.3.8 and 4.4.1 (Formal Positivestellensatz) of [7] imply that V contains a point α such that $\mathfrak{p}_\alpha = (0)$. Then $A(\alpha) = K$, so the equality $\rho_{U_i}(f_i)_\alpha = f(\alpha) = \rho_{U_j}(f_j)_\alpha \in A(\alpha)$ implies that $f_i = f_j$ in $A(\alpha) = K$, as desired.

Notation To simplify the notation, we will write A_i instead of A_{U_i}.

Definition 3.2.2 *A real geometric space is a locally ringed space (X, O_X) which admits an open covering $X = \bigcup_{i=1}^{s} Sper(A_i)$ such that each $(U_i, O_X|U_i)$ is isomorphic (as locally ringed space) to an affine real geometric space.*

Definition 3.2.3 *A real geometric surface is a real geometric space X where all A_i can be chosen to be regular 2-dimensional noetherian rings.*

Let k be a field and z an independent variable. Let A be a regular two-dimensional ring, x, y elements of A, $\mathfrak{p}$ a maximal ideal of A of height 2, containing x. Suppose given an isomorphism $\iota : \frac{A}{(x)} \tilde{\to} k[z]_\theta$ such that y mod (x) is sent to z and θ is a non-zero polynomial in z. Let $g = z^d + \bar{a}_1 z^{d-1} + \dots + \bar{a}_d$ denote the monic generator of the ideal $\iota\left(\frac{\mathfrak{p}}{(x)}\right)$. Let a_i be an element of the coset $\iota^{-1}(\bar{a}_i)$. Then $\left(x, y^d + a_1 y^{d-1} + \dots + a_d\right)$ is a set of generators of $\mathfrak{p}$; it induces a regular system of parameters of $A_\mathfrak{p}$.

Definition 3.2.4 *The pair $\left(x, y^d + a_1 y^{d-1} + \dots + a_d\right)$ will be called a **privileged system of parameters of** $A_\mathfrak{p}$ with respect to the ordered pair (x, y).*

Definition 3.2.5 *A marked real geometric surface is a real geometric surface X together with the following additional data:*

(1) A finite covering $X = \bigcup_{i=1}^{s} Sper(A_i)$ where each A_i is a regular 2-dimensional noetherian ring.

(2) For each i, a pair of elements $x_i, y_i \in A_i$ and a field k_i, which admits a total ordering.

*(3) A subset $\Delta_i \subset Sper\ A_i$, called the **privileged** subset of $Sper\ A_i$. Let z, w be independent variables. We require one of the following to hold:*

(a) There exists an irreducible polynomial $h \in k_i[w]$ and a homomorphism

$$\iota : A_i \to \frac{k_i[z, w]_{\theta_z \theta_w}}{(zh)},$$

where $\theta_z \in k_i[z, w] \setminus (z, h)$, $\theta_w \in k_i[w] \setminus (h)$, which maps x_i to z mod (zh), y_i to w mod (zh) such that Δ_i is the set of points of $Sper\ A_i$ defined by the vanishing of all the elements of $Ker\ \iota$ (in particular, $\Delta_i \cong Sper \frac{k_i[z, w]_{\theta_z \theta_w}}{(zh)}$);

(b) $\Delta_i = \{x_i = 0\}$*; there is an isomorphism* $\iota : \dfrac{A_i}{(x_i)} \to k_i[w]_{\theta_w}$*, where* θ_w *is a non-zero polynomial in* $k_i[w]$*, which sends* $y_i \mod (x_i)$ *to* w*; in particular,* $\Delta_i \cong$ *Sper* $k_i[w]_{\theta_w}$*;*

(c) $\Delta_i = \{x_i = y_i = 0\}$*; we have* $\dfrac{A_i}{(x_i, y_i)} \cong k_i$*; in particular,* $\Delta_i \cong$ *Sper* k_i*.*

(4) For each i *and each* $\alpha \in \{x_i = 0\} \subset \Delta_i$ *with ht* $\mathfrak{p}_\alpha = 2$*, a regular system of parameters of* $(A_i)_{\mathfrak{p}_\alpha}$*, privileged with respect to* (x_i, h) *in case (a) and with respect to* (x_i, y_i) *in case (b).*

(5) In case (a), for each i *and each* $\alpha \in \{h = 0\} \subset \Delta_i$ *with ht* $\mathfrak{p}_\alpha = 2$*, a regular system of parameters of* $(A_i)_{\mathfrak{p}_\alpha}$*, privileged with respect to* (h, x_i)*.*

Remark 3.2.6 *Let* A *be a regular 2-dimensional ring,* $\mathfrak{m}$ *a maximal ideal of* A *and* (x, y) *a regular system of parameters of* $A_{\mathfrak{m}}$*. Then Sper* A *is a marked real geometric surface.*

We now define the notion of blowing up of a real marked geometric surface. Let $X = \bigcup_i \text{Sper } A_i$ be a marked real geometric surface and take a point $\delta \in X$. Assume that δ belongs to the privileged set and $\text{ht}(\mathfrak{p}_{\delta,i}) = 2$ in every affine chart Sper A_i containing δ. We want to define the **blowing up of** X **along** δ. First consider the case $X = \text{Sper } A$. Let $x, y \in A$ and k be the pair of elements and the field appearing in the definition of marked real geometric surface.

Let (u, v) be the privileged system of regular parameters of $A_{\mathfrak{p}_\delta}$ given by the definition. It follows from definition that $(u, v) = (x, y)$ in Case (c), $u = x$ in Case (a) provided $\delta \in \{x = 0\}$ as well as in Case (b), and $u = h$ in Case (a) if $\delta \in \{h = 0\} \setminus \{x = 0\}$.

A **blowing up** of Sper A along $\mathfrak{p}_\delta$ (or, by abuse of language, blowing up along δ) is the marked real geometric surface X' defined as follows. As a topological space, we put $X' = \text{Sper } A'_1 \cup \text{Sper } A'_2$, where $A'_1 = A\left[\frac{v}{u}\right]$, $A'_2 = A\left[\frac{u}{v}\right]$ and

$$\text{Sper } A'_1 \cap \text{Sper } A'_2 = \text{Sper } A'_1 \setminus \left\{\frac{v}{u} = 0\right\} = \text{Sper } A'_2 \setminus \left\{\frac{u}{v} = 0\right\}.$$

We have a natural surjective morphism $\pi : X' \to \text{Sper } A$.

To define a structure of marked real geometric surface on X', we let the two elements required in Definition 3.2.5 (2) be $x'_1 = u, y'_1 = \frac{v}{u} \in A'_1$ for Sper A'_1 and $x'_2 = v, y'_2 = \frac{u}{v} \in A'_2$ for Sper A'_2. Below, for $q \in \{1, 2\}$, we denote the privileged set of A'_q by Δ'_q and the field required in the Definition 3.2.5 (2) for Sper A'_q by k'_q. We now define Δ'_q and k'_q in the different cases.

• If Case (c) holds for Sper A: let $k'_q = k$, for $q \in \{1, 2\}$. For Sper A'_1 the privileged set is $\Delta'_1 = \{x'_1 = 0\}$. The existence of a privileged regular system of parameters required by the Definition 3.2.5 comes from the isomorphism $\dfrac{A'_1}{(x'_1)} \cong k[y'_1]$. For Sper A'_2 the situation is entirely analogous.

• If Case (b) holds for Sper A : let $k'_1 = \kappa(\mathfrak{p}_\delta)$ and $\Delta'_1 = \{x'_1 = 0\}$. The existence of a privileged regular system of parameters required by the Definition 3.2.5 comes from $\dfrac{A'_1}{(x'_1)} \tilde{\to} k'_1[y'_1]$.

Let $k'_2 = k$ and $\Delta'_2 = \{x'_2 = 0\} \cup \{y'_2 = 0\}$. By the definition of privileged regular system of parameters of $A_{\mathfrak{p}_\delta}$, there is an irreducible polynomial $v_w \in k[w]$, relatively prime to θ_w such that $\iota(\frac{\mathfrak{p}_\delta}{(x)}) = (v_w)$. The existence of a privileged regular system of parameters at any point of Δ'_2, required by the Definition 3.2.5, comes from $\dfrac{A'_2}{(x'_2 y'_2)} \tilde{\to} \dfrac{k[w, y'_2]_{\theta_w}}{(v_w y'_2)}$.

• If Case (a) holds, there are three cases to consider :

(i) $\delta \in \{x = 0\} \setminus \{h = 0\}$, Δ'_q, k'_q, $q = 1, 2$, are given by the same formulas as in Case (b). Let $A'_3 = A_v$. The structure of marked real geometric surface on Sper A'_3 is induced from that of Sper A. We have $k'_3 = k$ and $\Delta'_3 = \{x = 0\} \cup \{h = 0\}$ and $\dfrac{A_v}{(xh)} \tilde{\to} \dfrac{k[z, w]_{v_w \theta_w \theta_z}}{(zh)}$.

(ii) $\delta \in \{h = 0\} \setminus \{x = 0\}$, let $k'_1 = \kappa(\mathfrak{p}_\delta)$ and $k'_2 = \frac{k[w]}{(h)}$, $\Delta'_1 = \{x'_1 = 0\}$, $\Delta'_2 = \{x'_2 = 0\} \cup \{y'_2 = 0\}$. By the definition of privileged regular system of parameters of $A_{\mathfrak{p}_\delta}$, there is a polynomial $v_z \in k[z, w]$, such that $\iota(\frac{\mathfrak{p}_\delta}{(xh)}) = \frac{(h, v_z)k[z, w]_{\theta_z\theta_w}}{(xh)}$. The existence of a privileged regular system of parameters comes from the isomorphisms $\frac{A'_1}{(x'_1)} \tilde{\to} k'_1[y'_1]$ and $\frac{A'_2}{(x'_2y'_2)} \tilde{\to} \frac{k'_2[z, y'_2]_{\theta_z}}{(v_zy'_2)}$.

Let $A'_3 = A_v$. The structure of marked real geometric surface on Sper A'_3 is induced from that of Sper A. We have $k'_3 = k$, $\Delta'_3 = \{x = 0\} \cup \{h = 0\}$ and $\frac{A_v}{(xh)} \tilde{\to} \frac{k[z, w]_{v_z\theta_w\theta_z}}{(zh)}$.

(iii) $\delta = \{h = 0\} \cap \{x = 0\}$, recall that $u = x, v = h$. Let $k'_1 = \kappa(\mathfrak{p}_\delta)$ and $k'_2 = k$. Let $\Delta'_q = \{x'_q = 0\} \cup \{y'_q = 0\}$, $q = 1, 2$. The existence of a privileged regular system of parameters comes from the isomorphisms $\frac{A'_1}{(x'_1y'_1)} \tilde{\to} \frac{k'_1[z, y'_1]_{\theta_z}}{(zy'_1)}$ and $\frac{A'_2}{(x'_2y'_2)} \tilde{\to} \frac{k[w, y'_2]_{\theta_w}}{(hy'_2)}$ (recall that in this case $h = x'_2$).

We then define the real marked geometric surface X' to be $X' = \bigcup\limits_{i=1}^{p}$ Sper A'_i where $p = 2$ in cases (b), (c) and (a) (iii) and $p = 3$ in cases (a) (i) and (ii).

Remark 3.2.7 *Note that Sper $A'_3 \subset$ Sper A'_i, $i = 1, 2$; but, in the applications, we need to have the set Δ'_3 defined by fixed elements x'_3, y'_3.*

Let $X = \bigcup\limits_{i=1}^{s}$ Sper A_i be a marked real geometric surface and $\delta \in X$ belonging to the privileged set and supported in a height 2 ideal $\mathfrak{p}_{\delta,i}$ in some affine chart Sper A_i.

If $\delta \notin$ Sper A_i, let $X'_i =$ Sper A_i with the identity map $X'_i \to$ Sper A_i. If $\delta \in$ Sper A_i, let $X'_i \to$ Sper A_i be the blowing up of Sper A_i along $\mathfrak{p}_{\delta,i}$. Let (u, v) be the regular system of parameters of $(A_i)_{\mathfrak{p}_{\delta,i}}$ given by the definition of real marked geometric surface. We have $X'_i = \bigcup_{j=1}^{p}$ Sper A'_{ji} where $p = 2$ or 3 as above.

The marked real geometric surfaces $X'_1, \dots, X'_s$ and the maps $X'_i \to$ Sper A_i glue together in a natural way to give a marked real geometric surface $X' = \bigcup\limits_{i=1}^{s} X'_i$ and the map $X' \to X$.

Definition 3.2.8 *We call X'* ***the blowing up of*** *X along δ or the point blowing up of X along δ. The point δ is called the* ***center*** *of this blowing up. If $X =$ Sper A, the blowing up of X along δ depends only on the ideal $\mathfrak{p}_\delta$ and not on the ordering $\le_\delta$, so we may speak also about blowing up along $\mathfrak{p}_\delta$.*

Definition 3.2.9 *Let α, δ be two distinct points of the real marked surface Sper A with*

$$ht(\mathfrak{p}_\delta) = 2.$$

Let $\pi : X' \to$ Sper A be a blowing up along δ. Let (u, v) be the given privileged system of parameters at δ. Since $\alpha \neq \delta$, $\{u, v\} \not\subset \mathfrak{p}_\alpha$. If $u \notin \mathfrak{p}_\alpha$, the ***strict transform*** *α' of α is defined as follows. Let $\mathfrak{p}_{\alpha'}$ be the strict transform of $\mathfrak{p}_\alpha$ in A'_1 and $\le_{\alpha'}$ be the order of $\kappa(\mathfrak{p}_{\alpha'})$ induced by $\le_\alpha$ via the natural isomorphism $\kappa(\mathfrak{p}_\alpha) \cong \kappa(\mathfrak{p}_{\alpha'})$. If $v \notin \mathfrak{p}_\alpha$, $\alpha' \in$ Sper A'_2 is defined similarly.*

On the way to prove the connectedness of C of (76), we will now prove a preliminary result on connectedness of a certain type of subsets (intervals) of the exceptional divisor on a suitable blowing up of Sper A.

Remark 3.2.10 *Fix an order on k. Let D be the set of points $\delta \in Sper(k[z])$ which induce the given order on k. Given two points $\delta_1 \neq \delta_2 \in D$ such that $ht(\mathfrak{p}_{\delta_i}) = 1$, we view δ_1, δ_2 as* *elements of the real closure $\overline{k}$ of k with respect to the given order. We may speak about the* ***interval*** $(\delta_1, \delta_2) = \{\delta \in D \mid \delta_1 < z(\delta) < \delta_2\}$. *If* $\mathfrak{p}_\delta = (0)$, *we compare* δ_i *and* $z(\delta)$ *via the natural embeddings* $k[z](\delta) \hookrightarrow \overline{k(z)}$ *and* $\overline{k} \subset \overline{k(z)}$.

Now, let $\mathfrak{m}$ *be an ideal of* A *with* $ht\ \mathfrak{m} = 2$ *and* $\frac{A}{\mathfrak{m}} = k$. *Given a blowing up along* $\mathfrak{m}$ *as above, consider the open set* $Sper(A[\frac{y}{x}])$. *The set of points* $\delta \in Sper(A[\frac{y}{x}])$ *such that* $x(\delta) = 0$ *and which induce the given order on* k *is homeomorphic to* D.

Finally, let X *be a real algebraic surface such that* $D \subset Sper\, k[z] \subset X$. *Let* $+\infty$ *denote the point of* D *with support* (0) *such that* $z(+\infty) > c$ *for all* $c \in k$. *Let* $\overline{\infty}$ *be the closed point of* X *such that* $\overline{\infty} \in \overline{\{+\infty\}}$. *Assume there is an open set* $Sper\, A_i \subset X$ *such that* $\mathfrak{p}_{\overline{\infty}}$ *in* A_i *has height 2. We extend the above notion of interval to include the case when* $\delta_2 = \overline{\infty}$ *with the obvious meaning assigned to* $[\delta_1, \overline{\infty}] = \bigcup\limits_{\delta > \delta_1} [\delta_1, \delta] \cup \{\overline{\infty}\}$, $(\delta_1, \overline{\infty}), \ldots$. *Similarly, we may take a closed point* $-\overline{\infty} \in \overline{\{-\infty\}}$. *As points of* X, *we have* $\overline{\infty} = -\overline{\infty}$. *However, our ordering on* D *provides us with a well defined notion of intervals of the form* $(-\overline{\infty}, \delta_1)$, $[-\overline{\infty}, \delta_1)$ *and so on.*

Lemma 3.2.11 *Let* D *be as in the remark before and* $\delta_1 < \delta_2 \in D$ *such that* $ht(\mathfrak{p}_{\delta_i}) = 1$. *The closed interval* $[\delta_1, \delta_2]$, *the semi-open interval* $[\delta_1, \delta_2)$ *and the open interval* (δ_1, δ_2) *are connected.*

Proof : We will prove it for the open case, the closed and the semi-open being similar. Let $k \hookrightarrow \overline{k}$ be the inclusion of k into its real closure determined by the given order. This map corresponds to a morphism $\mathrm{Sper}(\overline{k}[z]) \to \mathrm{Sper}(k[z])$ which induces a homeomorphism between D and $\mathrm{Sper}(\overline{k}[z])$ sending (δ_1, δ_2) to an interval $(\overline{\delta}_1, \overline{\delta}_2)$ where $\overline{\delta}_1, \overline{\delta}_2 \in \overline{k}$. It is well-known and easy to prove that such an interval is connected - in the spectral topology (see for instance [7]). $\square$

Remark 3.2.12 *Let* $\theta \in k[z]$ *be a non-zero polynomial. We have natural homeomorphisms* $Sper\, k[z]_\theta \tilde{\to} Sper\, k[z] \backslash \{\alpha_1, \ldots, \alpha_t\}$ *and* $\lambda : D \cap Sper\, k[z]_\theta \tilde{\to} D \backslash \{\alpha_1, \ldots, \alpha_t\}$ *where* $\{\alpha_1, \ldots, \alpha_t\}$ *is the set of points* $\alpha_i \in Sper\, k[z]$ *such that* $\theta \in \mathfrak{p}_{\alpha_i}$. *Let* $\delta_1, \delta_2 \in D$ *be as above. Assume that* $\alpha_i \notin (\delta_1, \delta_2)$ *for all* $i \in \{1, \ldots, t\}$. *Then* $\lambda((\delta_1, \delta_2))$ *is connected in* $D \setminus \{\alpha_1, \ldots, \alpha_t\}$.

Definition 3.2.13 *Let* $\mathfrak{m}$ *be a maximal ideal of* A *of height 2. Let* $X' \to Sper\, A$ *be the blowing up along* $\mathfrak{m}$. *Let* $\mathcal{E} = \{\epsilon \in Sper\, A \mid \mathfrak{p}_\epsilon = \mathfrak{m}\}$. *The sets* $\pi^{-1}(\epsilon)$, $\epsilon \in \mathcal{E}$ *are called the* ***components*** *of* $\pi^{-1}(\mathfrak{m})$.

Let $(A, \mathfrak{m}, k)$ be a regular 2-dimensional local ring and (x, y) a regular system of parameters. Now consider a sequence

$$X_t \overset{\pi_{t-1}}{\to} \cdots \overset{\pi_1}{\to} X_1 \overset{\pi_0}{\to} \mathrm{Sper}\ A \tag{109}$$

of point blowings up where the first blowing up $\pi_0 : X_1 \to \mathrm{Sper}\ A$ is the blowing up along $\mathfrak{m}$.

Fix a point $\epsilon \in \mathrm{Sper}\ A$ such that $\mathfrak{p}_\epsilon = \mathfrak{m}$ - this is equivalent to fixing a total ordering on k. For $q \in \{0, \ldots, t-1\}$, let $\eta_q \in X_q$ be the closed point, compatible with the given order, such that π_q is a blowing up along η_q.

For $i \in \{1, \ldots, t\}$, let $X_i = \bigcup\limits_{j=1}^{s_i} \mathrm{Sper}\ A_{ji}$ be the open affine covering in the definition of marked real geometric surface.

Let $\rho_i = \pi_0 \circ \ldots \circ \pi_{i-1} : X_i \to \mathrm{Sper}\ A$.

Remark 3.2.14 *The real geometric space* $\rho_i^{-1}(\mathfrak{m})$ *has the form* $\rho_i^{-1}(\mathfrak{m}) = \bigcup\limits_{\ell} Sper\, B_{i\ell}$ *with* $B_{i\ell} \cong k_{i\ell}[z_{i\ell}]$ *where* $k_{i\ell}$ *is a finite algebraic extension of* k *and* $z_{i\ell}$ *is an independant variable.*

Definition 3.2.15 *A subset* $E \subset \rho_i^{-1}(\epsilon)$ *is a* ***component*** *of* $\rho_i^{-1}(\epsilon)$ *if* E *is either a component of* $\pi_{i-1}^{-1}(\eta_{i-1})$ *or a strict transform of a component of* $\rho_{i-1}^{-1}(\epsilon)$ *when* $i > 1$.

Definition 3.2.16 *Let $\rho_i : X_i \to Sper(A)$. Fix a component $E \subset \rho_i^{-1}(\epsilon)$. Fix an index $j \in \{1, \dots, s_i\}$. A j-**distinguished** point of E is a point $\delta \in E$ such that either $\delta \not\in Sper\ A_{ji}$ or $\rho_i^{-1}(\{xy = 0\}) \supset \{x'y' = 0\}$ and $x'(\delta) = y'(\delta) = 0$ where $(x', y') \in A_{ji}$ is the privileged regular system of parameters at δ (in particular, the privileged set of Sper A_{ji} is given by $\{x' = 0\} \cup \{y' = 0\}$).*

A j-maximal interval I is a subset $I \subset E$ such that there exist j-distinguished points $\delta_1, \delta_2 \in E$, $\delta_1 \neq \delta_2$, such that

(1) $I = [\delta_1, \delta_2]$ and I is connected;

(2) There are no j-distinguished points in $I \setminus \{\delta_1, \delta_2\}$.

A maximal interval is an interval which is j-maximal for some j.

Remark 3.2.17 *Note that a j-maximal interval may contain a $\tilde{\jmath}$-distinguished point, where $j \neq \tilde{\jmath}$. This occurs if $[\delta_1, \delta_2]$ is a j-maximal interval, $\delta \in (\delta_1, \delta_2)$ and $\exists \tilde{\jmath} \in \{1, \dots, s_i\}$, $\tilde{\jmath} \neq j$, such that $(\delta_1, \delta_2) \cap Sper\, A_{\tilde{\jmath} i} \neq \emptyset$ and $\delta \not\in Sper\, A_{\tilde{\jmath} i}$.*

Proposition 3.2.18 *Fix a component $E \subset \rho_i^{-1}(\epsilon)$ and a maximal interval $[\delta_1, \delta_2] \subset E$. Take $q \in \{1\ , 2\}$. There exists $j \in \{1, \dots, s_i\}$ such that $[\delta_1, \delta_2]$ is j-maximal and letting $x_i, y_i \in A_{ji}$ be the elements given by Definition 3.2.5 we have:*

$(1)_i$ $[\delta_1, \delta_2] \setminus \{\delta_q\} \subset Sper\ A_{ji}$,

$(2)_i$ for all $\delta \in [\delta_1, \delta_2] \setminus \{\delta_q\}$ with $ht(\mathfrak{p}_\delta) = 2$, x_i is a part of the given privileged regular system of parameters of $(A_{ji})_{\mathfrak{p}_\delta}$,

$(3)_i$ $[\delta_1, \delta_2] \cap Sper\ A_{ji} = \left\{\eta \in Sper\ A_{ji}\ \middle|\ x_i(\eta) = 0 \text{ and } \overline{\delta_1} \leq y_i(\eta) \leq \overline{\delta_2}\right\}$ where

$$\overline{\delta_1}, \overline{\delta_2} \in \overline{k_{ji}} \cup \{-\overline{\infty}, \overline{\infty}\},$$

with the notation of Remark 3.2.10 and the proof of Lemma 3.2.11.

Proof: First, let $i = 1$. We have $X_1 = \text{Sper } A\left[\frac{y}{x}\right] \cup \text{Sper } A\left[\frac{x}{y}\right]$. Denote $A\left[\frac{y}{x}\right]$ by A_{11} and $A\left[\frac{x}{y}\right]$ by A_{21}. Let $x_1 = x, y_1 = \frac{y}{x}$. Fixing the component E is equivalent to fixing a total order on k; this data is already given. We have

$$E \cap \text{Sper } A[y_1] \subset \text{Sper } k[y_1].$$

Let the notation be as in Remark 3.2.10 with y_1 playing the role of z.

There are exactly two maximal intervals $[0, \overline{\infty}]$ and $[-\overline{\infty}, 0]$. Say, for example, $I = [0, \overline{\infty}]$, $q = 2$, then $j = 1$ satisfies the conclusion of the Proposition. And similarly for the other three cases.

Now take $i \geq 2$ and suppose the result true for $i - 1$. Let $\delta_{p,i-1} = \pi_{i-1}(\delta_p)$, $p = 1, 2$. Let η_{i-1} be the center of the blowing up π_{i-1}. First, assume that

$$E \subset \pi_{i-1}^{-1}(\eta_{i-1}). \tag{110}$$

Take $\tilde{\jmath} \in \{1, \dots, s_{i-1}\}$ such that η_{i-1} belongs to the privileged set of Sper $A_{\tilde{\jmath}, i-1}$. Let (u, v) be the given privileged regular system of parameters at η_{i-1}. If j is such that $(\delta_1, \delta_2) \subset \text{Sper } A_{ji}$ then A_{ji} is one of $A_{\tilde{\jmath}, i-1}[\frac{u}{v}]$ or $A_{\tilde{\jmath}, i-1}[\frac{v}{u}]$; pick one of these two possible choices j such that $[\delta_1, \delta_2]$ is j-maximal. In this case $(1)_i$ is equivalent to saying that

$$[\delta_1, \delta_2] \neq [-\overline{\infty}, \overline{\infty}]. \tag{111}$$

Now, if we had $[\delta_1, \delta_2] = [-\overline{\infty}, \overline{\infty}]$, the point $x_i = y_i = 0$ would be a distinguished point in (δ_1, δ_2) (by definition of distinguished point). This is a contradiction and $(1)_i$ is proved in the case when (110). $(2)_i$ and $(3)_i$ of the Proposition follow immediately from the definition of marked real geometric surface.

¿From now on, assume that

$$E \not\subset \pi_{i-1}^{-1}(\eta_{i-1}). \tag{112}$$

Note that since $[\delta_1, \delta_2]$ is a maximal interval of E, $[\delta_{1,i-1}, \delta_{2,i-1}]$ is a maximal interval of $\pi_{i-1}(E)$. So, by the induction hypothesis, the Proposition holds for $[\delta_{1,i-1}, \delta_{2,i-1}] \subset \pi_{i-1}(E)$. Take $\tilde{\jmath} \in \{1, \ldots, s_{i-1}\}$ which satisfies the conclusion of the Proposition with i replaced by $i-1$ (in particular, $[\delta_{1,i-1}, \delta_{2,i-1}]$ is $\tilde{\jmath}$-maximal).

If $\eta_{i-1} \notin$ Sper $A_{\tilde{\jmath},i-1}$, take $j \in \{1, \ldots, s_i\}$ such that $A_{ji} = A_{\tilde{\jmath},i-1}$. This j satisfies the conclusion of the Proposition.

Next assume that $\eta_{i-1} \in$ Sper $A_{\tilde{\jmath},i-1}$. Take the elements $u, v \in A_{\tilde{\jmath},i-1}$ which induce the privileged regular system of parameters at η_{i-1}, given by the definition of marked real geometric surface.

If $\eta_{i-1} \notin \pi_{i-1}([\delta_1, \delta_2])$, take j such that $A_{ji} = A_{\tilde{\jmath},i-1}$.

¿From now on, assume that $\eta_{i-1} \in \pi_{i-1}([\delta_1, \delta_2]) \cap$ Sper $A_{\tilde{\jmath},i-1}$. Then

$$\eta_{i-1} \in \{\delta_{1,i-1}, \delta_{2,i-1}\} :$$

if not, $\pi_{i-1}^{-1}(\eta_{i-1}) \cap (\delta_1, \delta_2)$ would be a j-distinguished point in (δ_1, δ_2), which is impossible. The intersection is taken as subsets of the topological space X_i; if $\eta_{i-1} \notin \{\delta_{1,i-1}, \delta_{2,i-1}\}$, this intersection is not empty and consists of a single point. Let $j \in \{1, \ldots, s_i\}$ be such that A_{ji} is one of $A_{\tilde{\jmath},i-1}[\frac{u}{v}]$ or $A_{\tilde{\jmath},i-1}[\frac{v}{u}]$; pick one of these two possible choices j such that $[\delta_1, \delta_2]$ is j-maximal.

In all the cases the index j chosen in this way satisfies the conclusion of the Proposition. □

3.3 A proof of the Pierce-Birkhoff conjecture for regular 2-dimensional rings.

Let A be a 2-dimensional regular local ring, ν a valuation on A.imensional ring. In this section, we prove that A is a Pierce-Birkhoff ring ([26]). Our proof is based on Madden's unpublished preprint ([27]), but there are some differences. Here, we have tried to present a proof which should be a pattern for a general proof of the conjecture in any dimension. 1

Theorem 3.3.1 *Let A be a 2-dimensional regular ring, then A is a Pierce-Birkhoff ring.*

Actually, we prove that A satisfies the Definable Connectedness Conjecture and also, in the special case where A is excellent, the Connectedness Conjecture.

We start with some results which do not assume that A is excellent and which are needed in the proof of both of the above versions of the Connectedness Conjecture. Let $\alpha, \beta \in$ Sper A. By Remark 0.1.10, we may assume that neither of α, β is a specialization of the other.

There are two possibilities : either $ht(<\alpha, \beta>) = 1$ or $ht(<\alpha, \beta>) = 2$.

3.3.1 The case of height 1.

Let δ be the most general common specialization of α and β and let $\mathfrak{p} = \sqrt{<\alpha, \beta>}$ be the support of δ. Then $A_\mathfrak{p}$ is a discrete valuation ring; take an element $t \in A$ whose image in $A_\mathfrak{p}$ is a regular parameter of $A_\mathfrak{p}$. Since $ht(\mathfrak{p}) = 1$ and neither of α, β is a specialization of the other, we have $\mathfrak{p}_\alpha = \mathfrak{p}_\beta = (0)$. There are only two orders on A which induce the given order on $A/\mathfrak{p}$: one with $t > 0$ and one with $t < 0$. Since $\alpha \neq \beta$, $<\alpha, \beta>= \mathfrak{p}$: of course, any element g of $\mathfrak{p}$ can be written as $g = t^\gamma \frac{a}{b}$, $a, b \notin \mathfrak{p}$. As $t \in <\alpha, \beta>$, if $\gamma \geq 2$, $\nu_\alpha(g) = \nu_\beta(g) > \nu_\alpha(t)$ so $g \in <\alpha, \beta>$ and if $\gamma = 1$, g changes sign between α and β, so again $g \in <\alpha, \beta>$.

Now let $f_1, \ldots, f_r \notin <\alpha, \beta>= \mathfrak{p}$, so $f_i(\delta) \neq 0$ for $i \in \{1, \ldots, r\}$. As $\delta \in \overline{\{\alpha\}}$ and $\delta \in \overline{\{\beta\}}$, we conclude that α and β belong to the same connected component of Sper $A \backslash \{f_1 \cdots f_r = 0\}$.

3.3.2 The case of height 2.

Now assume

$ht(<\alpha,\beta>)=2$, that is $\mathfrak{m}=\sqrt{<\alpha,\beta>}$ is maximal. By Proposition 2.1.2, replacing A by $A_{\mathfrak{m}}$ does not change the problem, so we may assume that A is local with maximal ideal $\mathfrak{m}$.

Let $g\in\mathbb{N}$ be such that $Q_1,\dots,Q_g\notin <\alpha,\beta>$, $Q_{g+1}\in <\alpha,\beta>$ be the approximate roots common to ν_α and ν_β as in section 3.1.

Let (x,y) be a regular system of parameters of A such that $\nu_\alpha(x)=\nu_\alpha(\mathfrak{m})$ and $\nu_\beta(x)=\nu_\beta(\mathfrak{m})$.

Let $\pi:A\to A'$ be a local blowing up with respect to ν_α and denote by k' the residue field of A'. Recall from ([45], Appendix 5) that the **weak transform** $I'\subset A'$ of an ideal $I\subset A$ is defined by $I'=x^{-a}IA'$ where $a=\nu_{\mathfrak{m}}(I)$.

Proposition 3.3.2 *We assume that π is also a local blowing up with respect to ν_β. Let α' and β' be the strict transforms of α and β. Then the separating ideal $<\alpha',\beta'>$ is equal to the weak transform of $<\alpha,\beta>$.*

Proof : Since by hypothesis, α', β' are both centered at a maximal ideal $\mathfrak{m}'$, we have $<\alpha,\beta>\subsetneq\mathfrak{m}$. In particular, $x\notin <\alpha,\beta>$, hence x does not change sign between α and β. Then $f\in A$ changes sign between α and β if and only if $x^{-a}f$ changes sign between α' and β'.

Since $<\alpha,\beta>$ is generated by elements changing sign between α and β, its weak transform is generated by elements which change sign between α' and β'; hence the weak transform of $<\alpha,\beta>$ is contained in $<\alpha',\beta'>$.

To prove the opposite inclusion, let $I'=<\alpha',\beta'>$ and let I be the inverse transform of I', that is the unique complete ideal of A whose weak transform is I' ([45], Appendix 5, p. 388). It remains to prove that $I\subseteq<\alpha,\beta>$.

In order to do this, it suffices to find an element $z\in I$ which changes sign between α and β and such that $\nu_\alpha(z)=\nu_\alpha(I)$.

Let J_+ be the greatest ν_α-ideal of A' whose ν_α-value is strictly greater than $\nu_\alpha(I)$. Note that $\dfrac{IA'}{J_+\cap IA'}$ is a k'-vector space. Let $b_1,\dots,b_\ell$, $b_j=\prod_{r=1}^i Q_r^{\gamma_{jr}}$, where i is the maximal index of the approximate roots Q_s involved, be a set of elements of I which induces a basis of $\dfrac{IA'}{J_+\cap IA'}$, each monomial being standard. Moreover, since x divides y in A', if $\nu_\alpha(x)=\nu_\alpha(y)$, we may assume $\gamma_{j2}=0$ for all j and b_1 is the unique monomial which maximizes the vector $(\gamma_{i1},\gamma_{i-1,1},\dots,\gamma_{31})$ in the lexicographical ordering.

Let $a=\nu_{\mathfrak{m}}(I)$. Let $\tilde z\in I'$ be such that $\nu_\alpha(\tilde z)=\nu_\alpha(I')$ and $\tilde z$ changes sign between α' and β'. Let $z^\dagger=x^a\tilde z$. Then $z^\dagger\in IA'$ and $\nu_\alpha(z^\dagger)=\nu_\alpha(IA')=\nu_\alpha(I)$. Write $z^\dagger=\sum_{j=1}^\ell z_jb_j$. We may assume $z_1=1$. Denote by $\overline{z}_j$ the image of z_j in the residue field k'.

First, suppose $\nu_\alpha(x)<\nu_\alpha(y)$. Then $k'=k$. For each $j\in\{1,\dots,\ell\}$, let w_j be a representative of $\overline{z}_j$ in A. Put $z=\sum_{j=1}^\ell w_jb_j$.

Next, suppose $\nu_\alpha(x)=\nu_\alpha(y)$, since b_1 is the unique monomial which maximizes the vector $(\gamma_{i1},\gamma_{i-1,1},\dots,\gamma_{31})$, by the corollary (3.1.11), we have, for $j\geq 2$,

$$\nu_{\mathfrak{m}}\left(\prod_{r=3}^{i}Q_r^{\gamma_{jr}}\right)\leq\nu_{\mathfrak{m}}\left(\prod_{r=3}^{i}Q_r^{\gamma_{1r}}\right)-n\leq a-n.$$

Write $z^\dagger=b_1+\sum_{j=2}^\ell(z_jx^n)(x^{-n}b_j)$. Write $\overline{z}_j=\displaystyle\sum_{t=0}^{n-1}c_t\overline{\left(\frac{y}{x}\right)}^t$ where $c_t\in k$. So letting a_t be an element of A such that $\overline{a}_t=c_t$ and $v_j\in A$ be the element $v_j=\sum_{t=0}^{n-1}a_ty^tx^{n-t}$, we have

$$\nu_\alpha(v_j-z_jx^n)>n\nu_\alpha(x). \tag{113}$$

Lemma 3.3.3 *For $j \geq 2$, $x^{-n}b_j \in A$.*

Proof of lemma : By Corollary 3.1.10, $\nu_{\mathfrak{m}}(b_j) > \nu_{\mathfrak{m}}(b_1)$ and by Corollary 3.1.11,

$$\nu_{\mathfrak{m}}\left(\prod_{r=3}^{i} Q_r^{\gamma_{jr}}\right) \leq \nu_{\mathfrak{m}}\left(\prod_{r=3}^{i} Q_r^{\gamma_{1r}}\right) - n.$$

Now $\gamma_{j1} = \nu_{\mathfrak{m}}(b_j) - \nu_{\mathfrak{m}}\left(\prod\limits_{r=3}^{i} Q_r^{\gamma_{jr}}\right) > \nu_{\mathfrak{m}}(b_1) - \nu_{\mathfrak{m}}\left(\prod\limits_{r=3}^{i} Q_r^{\gamma_{1r}}\right) + n \geq n.$ $\Box$

Put $z = b_1 + \sum_{j=2}^{\ell} v_j(x^{-n}b_j)$ We have $z \in IA' \cap A = I$ (because I is a contracted ideal).

In both cases, $\nu_\alpha(x) = \nu_\alpha(y)$ and $\nu_\alpha(x) < \nu_\alpha(y)$, since $z^\dagger$ changes sign between α' and β' and in view of (113), z changes sign between α and β. This ends the proof of the proposition. $\Box$

Remark 3.3.4 *If $B = R[x,y]$. Let*

$$B \to R[x_1, y_1] \to \cdots \to R[x_\ell, y_\ell]$$

be a sequence of blowings up induced by (102), where we take $I =< \alpha, \beta >$. Let C_ℓ be the preimage of C (see (76)) in Sper $R[x_\ell, y_\ell]$. By proposition (3.1.7), there exist monomials $\omega_1, \dots, \omega_s, \epsilon_1, \dots, \epsilon_s, \theta_1, \dots, \theta_t,\ \lambda_1, \dots,\ \lambda_t$ in x_ℓ, y_ℓ such that

$$C_\ell = \left\{ \delta \in Sper\ R[x_\ell, y_\ell] \;\middle|\; \begin{array}{c} \nu_\delta(\omega_k) < \nu_\delta(\epsilon_k),\ k \in \{1, \dots, s\} \\ \nu_\delta(\theta_j) = \nu_\delta(\lambda_j),\ j \in \{1, \dots, t\} \\ sgn_\delta(x_\ell) = sgn_\alpha(x_\ell), sgn_\delta(y_\ell) = sgn_\alpha(y_\ell) \end{array} \right\}.$$

By connectedness theorem ([21]), C_ℓ is connected, hence so is C. This completes the proof of the Connectedness Conjecture for $R[x,y]$ and so provides a new proof of the classical Pierce-Birkhoff Conjecture in dimension 2.

Let A be a regular 2-dimensional local ring with regular parameters (x, y). Consider the set C' defined by the inequalities (77)

$$\left|\sum_{j=1}^{m_i} b_{ji}\mathbf{Q}^{\theta_{ji}}\right| >_\delta n_i|\mathbf{Q}^{\epsilon_{j'i}}|\ \forall i \in \{1, \dots, r\},\ \forall j' \in \{1, \dots, n_i\} \tag{114}$$

and the two sign conditions appearing in (76).

Consider the sequence (102) of local blowings up with $I =< \alpha, \beta >$. Let C'_ℓ be the preimage of C' in Sper A_ℓ. Rather than prove connectedness of C'_ℓ, we will prove that $\alpha^{(\ell)}$ and $\beta^{(\ell)}$ lie in the same connected component of C'_ℓ; this will imply that α and β lie in the same connected component of C'. Let ϵ denote the common specialization of $\alpha^{(\ell)}$ and $\beta^{(\ell)}$. By definition of (102), we have $\mathfrak{p}_\epsilon = \mathfrak{m}_\ell$. Let U be the subset of C'_ℓ consisting of all the generizations of ϵ lying in C'_ℓ. It is sufficient to prove that $\alpha^{(\ell)}$ and $\beta^{(\ell)}$ lie in the same connected component of U.

There are two cases to consider.

Case 1. Only one component of the exceptional divisor (that is the inverse image $\rho_{\ell-1}^{-1}(\mathfrak{m})$) passes through η_ℓ.

Case 2. Two components of the exceptional divisor pass through η_ℓ.

Let (x_ℓ, y_ℓ) be a regular system of parameters of $(A_\ell)_{\mathfrak{m}_\ell}$ such that the local equation of the exceptional divisor at η_ℓ is $x_\ell = 0$ in case 1 and $x_\ell y_\ell = 0$ in case 2.

By Zariski's theory of complete ideals, for any $f \in A\setminus < \alpha, \beta >$, the strict transform of f in A_ℓ is a unit. In other words, f has the form $f = x_\ell^n v$ in case 1 (resp. $f = x_\ell^n y_\ell^m v$ in case 2) where v denotes a unit in $(A_\ell)_{\mathfrak{m}_\ell}$.

The inequalities (114), appearing in the definition of C', hold on all of U. The set U is defined inside the set of generizations of ϵ in Sper A_ℓ either by specifying sgn(x_ℓ) or by specifying both sgn(x_ℓ) and sgn(y_ℓ).

Lemma 3.3.5 *Let E be an irreducible component of the exceptional divisor passing through η_ℓ, defined by $x_\ell = 0$. There exists $f \in A \setminus <\alpha, \beta>$ such that $f = x_\ell^n v$, v is a unit of $(A_\ell)_{\mathfrak{m}_\ell}$ and n is odd.*

Proof : Let $j \in \{1, \dots, \ell - 1\}$ be such that E is the strict transform in X_ℓ of $\pi_{j-1}^{-1}(\eta_{j-1})$. Let ν_j be the divisorial valuation corresponding to E; this valuation is defined as follows : for each $f \in A_\ell$, write $f = x_\ell^n g$ such that $x_\ell \nmid g$ in $(A_\ell)_{\mathfrak{m}_\ell}$, then $\nu_j(f) = n$.

Let $\mathfrak{m} = \mathfrak{p}_0 \supset \cdots \supset \mathfrak{p}_j$ be the complete list of simple ν_j-ideals given by Zariski's theory of complete ideals. Note that, since $j < \ell$, $\mathfrak{p}_j \supset <\alpha, \beta>$.

It follows from Zariski's factorization theorem for complete ideals that $\nu_j(A \setminus \{0\})$ is generated by $\nu_j(\mathfrak{p}_0), \dots, \nu_j(\mathfrak{p}_j)$. Since the value group of ν_j is $\mathbb{Z}$, the semigroup $\nu_j(A \setminus \{0\})$ contains all the sufficiently large integers. Hence one of $\nu_j(\mathfrak{p}_0), \dots, \nu_j(\mathfrak{p}_j)$ is odd. $\square$

The lemma shows that x_ℓ does not change sign between $\alpha^{(\ell)}$ and $\beta^{(\ell)}$ in Case 1 (resp. neither x_ℓ nor y_ℓ change sign between $\alpha^{(\ell)}$ and $\beta^{(\ell)}$ in Case 2).

Let

$$\tilde{U} = \left\{ \delta \in \text{Sper } A_\ell \;\middle|\; sgn(x_\ell(\delta)) = sgn(x_\ell(\alpha)), \epsilon \in \overline{\{\delta\}} \right\}$$

in Case 1 and

$$\tilde{U} = \left\{ \delta \in \text{Sper } A_\ell \;\middle|\; sgn(x_\ell(\delta)) = sgn(x_\ell(\alpha)), sgn(y_\ell(\delta)) = sgn(y_\ell(\alpha)), \epsilon \in \overline{\{\delta\}} \right\}$$

in Case 2. The above reasoning shows that $\alpha^{(\ell)}, \beta^{(\ell)} \in \tilde{U} \subset U$.

To prove the Definable Connectedness Conjecture (resp. the Connectedness Conjecture for excellent A), it remains to prove the definable connectedness of $\tilde{U}$ (resp. connectedness of $\tilde{U}$ whenever A is excellent).

We are now ready to prove the above two versions of the Connectedness Conjecture.

3.4 Proof of the Connectedness Conjecture in the case of an excellent regular 2-dimensional ring.

Theorem 3.4.1 *Let A be an excellent regular local 2-dimensional ring. Let $C \subset$ Sper A be the subset satisfying the conditions of (76). Then α and β belong to the same connected component of C.*

Proof: Let ϵ, ℓ and $\tilde{U}$ as above. By the above considerations, it is sufficient to prove that $\tilde{U}$ is connected. Thus it remains to prove the following lemma.

Lemma 3.4.2 *Let A be an excellent regular n-dimensional local ring, $x_1, \dots, x_n$ regular parameters of A. Fix a subset $T \subset \{1, \dots, n\}$ and let $D = \{\delta \in$ Sper $A \mid x_i(\delta) > 0,\ i \in T$ and $\epsilon \in \overline{\{\delta\}}\}$. Then D is connected.*

Proof: The point ϵ determines an order on k. Let R denote the real closure of k relative to this order. Consider the natural homomorphisms

$$A \to \hat{A} = k[[X_1, \dots, X_n]] \xrightarrow{\sigma} R[[X_1, \dots, X_n]] \tag{115}$$

where σ is induced by ϵ.

Let $\hat{\epsilon}$ denote the point of Sper $\hat{A}$ such that $\mathfrak{p}_{\hat{\epsilon}} = (X_1, \dots, X_n)$ and $\leq_{\hat{\epsilon}}$ is the total ordering of k given by ϵ.

Following ([3], proposition 8.6), D is connected if and only if

$$\hat{D} = \{\delta \in \text{Sper } k[[X_1, \dots, X_n]] \mid X_i(\delta) > 0,\ i \in T, \hat{\epsilon} \in \overline{\{\delta\}}\}$$

is connected (this is where we are using the fact that A is excellent). Moreover, $\hat{D}$ is the image of

$$\tilde{D} = \{\delta \in \text{Sper } R[[X_1, \dots, X_n]] \mid X_i(\delta) > 0,\ i \in T\}$$

under the natural map induced by σ

$$\text{Sper } R[[X_1, \dots, X_n]] \to \text{Sper } k[[X_1, \dots, X_n]].$$

Thus it suffices to prove that $\tilde{D}$ is connected.

By ([3], proposition 8.6), $\tilde{D}$ is connected if and only if the set

$$D^\dagger = \{\delta \in \text{Sper } R[X_1, \dots, X_n]_{(X_1,\dots,X_n)} \mid X_i(\delta) > 0,\ i \in T,\ \delta \text{ is centered at } (X_1, \dots, X_n)\}$$

is connected.

We have the following natural homomorphisms

$$\begin{array}{ccc} R[X_1,\dots,X_n] & \xrightarrow{\phi} & R[X_1,\dots,X_n]_{X_1\cdots X_n} \\ \big\downarrow{\scriptstyle\psi} & & \\ R[X_1,\dots,X_n]_{(X_1,\dots,X_n)} & & \end{array}$$

and the corresponding maps of real spectra

$$\begin{array}{ccc} \text{Sper } R[X_1,\dots,X_n]_{X_1,\dots,X_n} & \xrightarrow{\phi^*} & \text{Sper } R[X_1,\dots,X_n] \\ & & \big\uparrow{\scriptstyle\psi^*} \\ & & \text{Sper } R[X_1,\dots,X_n]_{(X_1,\dots,X_n)} \end{array} .$$

Define

$$D_0 = \{\delta \in \text{Sper } R[X_1, \dots, X_n] \mid X_i(\delta) > 0,\ i \in T,\ \delta \text{ is centered at } (X_1, \dots, X_n)\}$$

and

$$D_{loc} = \{\delta \in \text{Sper } R[X_1, \dots, X_n]_{X_1\cdots X_n} \mid X_i(\delta) > 0,\ i \in T,\ \phi^*(\delta) \text{ is centered at } (X_1, \dots, X_n)\}.$$

Now the maps ϕ^* and ψ^* induce homeomorphisms

$$\phi^*|_{D_{loc}} : D_{loc} \cong D_0 \quad \text{and} \tag{116}$$

$$\psi^*|_{D^\dagger} : D^\dagger \cong D_0. \tag{117}$$

Thus it suffices to prove that D_{loc} is connected. But

$$D_{loc} = \bigcap_{N\in\mathbb{N}} D_N$$

where

$$D_N = \left\{\delta \in \text{Sper } R[X_1, \dots, X_n]_{X_1\cdots X_n} \;\middle|\; \frac{1}{N} \geq X_i(\delta) \geq 0,\ i \in T\right\}.$$

By Proposition 7.5.1. of [7], each D_N is a non-empty closed connected subset of Sper $R[X_1, \dots, X_n]_{X_1\cdots X_n}$, hence D_{loc} is connected by ([21], lemma 7.1). $\square$

The lemma proves that any "quadrant" is connected, $\tilde{U}$ is a quadrant, hence it is connected. This completes the proof of the Connectedness Conjecture for any excellent 2-dimensional ring A.

Remark 3.4.3 *The above proof is a special case of the following general principle. Let A be an excellent regular local ring with regular parameters $x = (x_1, \dots, x_n)$ whose residue field k is equipped with a total ordering. Let R be the real closure of k. We have natural morphisms*

$$\begin{tikzcd} \mathit{Sper}\ A & \mathit{Sper}\ R[[X_1, \dots, X_n]] \arrow[l, "\phi"'] \arrow[d, "\pi"'] \\ & \mathit{Sper}\ R[X_1, \dots, X_n]_{(X_1, \dots, X_n)} \end{tikzcd}$$

Let $D \subset$ Sper A be a constructible set such that all the elements of A appearing in the definition of D belong to $A \cap R[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$. Let $\hat{D} = \phi^{-1}(D)$, let U be the subset of all points of Sper $R[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$ centered at the origin. Let D_{pol} be the subset of U defined by the same formulae as D. By ([3], proposition 8.6), to show that D is connected, it is enough to prove that D_{pol} is connected.

In many cases, this principle applies also to nested intersection $D = \bigcap\limits_{N \in \mathbb{N}} D_N$ of constructible sets defined by elements of $A \cap R[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$.

This allows to transpose all the results of ([21]) from the case of polynomial rings to that of arbitrary excellent regular local rings.

3.5 Proof of the Definable Connectedness Conjecture for regular 2-dimensional local rings.

Next we prove the Definable Connectedness Conjecture, hence the Pierce-Birkhoff Conjecture, without the excellence hypothesis on A.

Theorem 3.5.1 *Let $(A, \mathfrak{m}, k)$ be a regular 2-dimensional local ring, (x, y) a regular system of parameters of A. The sets*

$$U = \{\delta \in \mathit{Sper}\ A \mid x(\delta) > 0, \epsilon \in \overline{\{\delta\}}\} \tag{118}$$
$$V = \{\delta \in \mathit{Sper}\ A \mid x(\delta) > 0, y(\delta) > 0, \epsilon \in \overline{\{\delta\}}\} \tag{119}$$

are definably connected.

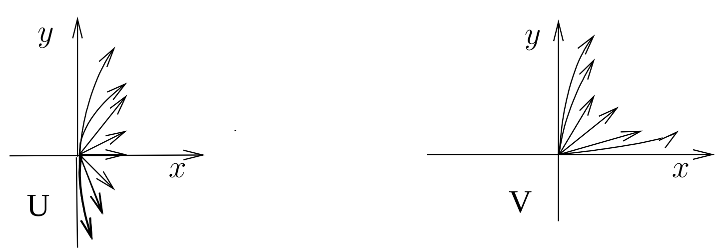

Figure 1: The sets U and V

Proof : We argue by contradiction. Let Ω be either U or V. Write $\Omega = F \coprod G$, $F = \bigcup F_i$, $G = \bigcup G_i$ where $\{F_i\}$, $\{G_i\}$ are finite collections of basic open sets. Each F_i and G_i is defined by finitely many inequalities of the form $g > 0$, $g \in A$. Let $g_3, \dots, g_r \in A$ be the list of elements of A, appearing in the definition of all of F_i and G_i and let $g_1 = x$, $g_2 = y$. A proof of the Theorem will be given after a few auxiliary definitions and results.

Let Sper $A \leftarrow X_1 \leftarrow \cdots \leftarrow X_t$ be a sequence of point blowings up. Let $X_t = \bigcup\limits_{j=1}^{s}$ Sper A_{jt} be the open covering of X_t, given by the definition of real geometric surface.

Definition 3.5.2 *We say that a collection $\{h_1,\dots,h_r\}$ of elements of A are simultaneously locally monomial in X_t if for all $j \in \{1,\dots,s\}$ and any maximal ideal $\mathfrak{m}' \subset A_{jt}$, there exists a regular system of parameters (x',y') of $A' := (A_{jt})_{\mathfrak{m}'}$ such that all of $h_1,\dots,h_r$ are monomials in (x',y') multiplied by units of $(A_{jt})_{\mathfrak{m}'}$.*

Let $g_1,\dots,g_r \in A$ be as above. By standard results on resolution of singularities, there exists a sequence $\operatorname{Sper} A \leftarrow X_1 \leftarrow \cdots \leftarrow X_t$ of point blowings up such that $g_1,\dots,g_r$ are simultaneously locally monomial in X_t. Denote by $\rho_t : X_t \to \operatorname{Sper} A$ the composition of all the morphisms in that sequence (with the notations following (109)).

Let $\Omega^{(t)} = \rho_t^{-1}(\Omega)$, $F^{(t)} = \rho_t^{-1}(F)$, $G^{(t)} = \rho_t^{-1}(G)$, $U^{(t)} = \rho_t^{-1}(U)$.

Take a point $\delta \in \rho_t^{-1}(\epsilon)$, let $A', \mathfrak{m}', x', y', A_{jt}$ be as in the definition of simultaneously locally monomial.

Definition 3.5.3 *We say that δ is a special point of $\rho_t^{-1}(\epsilon)$ if $ht(\mathfrak{p}_\delta) = 2$ and*

$$\{x'y' = 0\} = \rho_t^{-1}(\epsilon) \cup \{g_1 \cdots g_r = 0\}$$

locally near δ.

Given a special point $\delta \in \rho_t^{-1}(\epsilon)$ and (u',v') a regular system of parameters at δ, let

$$C(\delta,u',v') = \{\gamma \in X_t \mid u'(\gamma) > 0,\ v'(\gamma) > 0,\ \delta \in \overline{\{\gamma\}}\}.$$

Lemma 3.5.4 *Take a point $\xi \in \rho_t^{-1}(\epsilon)$, not lying on the strict transform of $\{x = 0\}$. Take $j \in \{1,\dots,s_i\}$ such that $\rho_t^{-1}(\epsilon)$ is contained in the privileged set of Sper A_{jt} near ξ. Let $x_{jt}, y_{jt} \in A_{jt}$ be the elements given in Definition 3.2.5. Assume that the privileged set is given by $\{x_{jt} = 0\}$ and is homeomorphic to Sper $k'[z]_{\theta_z}$, where θ_z is a non-zero polynomial, with k' finite over k and that $ht(\mathfrak{p}_\xi)$=2. Let (x',y') be as in the definition of simultaneously locally monomial where we take $\mathfrak{m}' = \mathfrak{p}_\xi$ (we may assume $x' = x_{jt}$). We view k' as an ordered field via the inclusion $k' \subset A_{jt}(\xi)$. Let*

$$E = \{\delta \in \text{Sper } A_{jt} \mid x_{jt}(\delta) = 0 \text{ and } k' \subset A_{jt}(\delta) \text{ is an inclusion of ordered fields }\}.$$

Take special points $\delta_1, \delta_2 \in E$ such that the intervals (δ_1,ξ) and (ξ,δ_2) are connected and contain no special points.

For $i \in \{1,2\}$, let (x',v_i') be a regular system of parameters at δ_i such that $\{v_i' > 0\} \cap (\delta_1,\delta_2) \neq \emptyset$.

Then the set

$$D(\delta_1,\delta_2) = C(\delta_1,x',v_1') \cup C(\delta_2,x',v_2') \cup \{\delta \in U^{(t)} \mid x'(\delta) > 0,\ \overline{\{\delta\}} \cap (\delta_1,\delta_2) \neq \emptyset\}$$

is contained either in $F^{(t)}$ or in $G^{(t)}$.

Proof : First, assume ξ is not special. Then there are no special points in (δ_1,δ_2). Let $F_\dagger = \overline{F^{(t)}} \cap [\delta_1,\delta_2]$ and $G_\dagger = \overline{G^{(t)}} \cap [\delta_1,\delta_2]$. Then $F_\dagger$, $G_\dagger$ are relatively closed in $[\delta_1,\delta_2]$ and $[\delta_1,\delta_2]$ is connected (Lemma 3.2.11) , so $F_\dagger \cap G_\dagger \neq \emptyset$.

Take a point $\eta \in F_\dagger \cap G_\dagger$. Replacing η by its specialization, we may assume that $ht(\mathfrak{p}_\eta) = 2$. For each $i \in \{1,\dots,r\}$, locally near η, write $g_i = x'^a g_i'$ if $\eta \notin \{\delta_1,\delta_2\}$ and $g_i = x'^a y'^b g_i'$ if $\eta = \delta_\ell$, $\ell \in \{1,2\}$ with $y' = v_i'$, where, in both cases, g_i' is invertible locally near η.

Take an open set W, containing η, such that for all $\delta \in W$ and all $i \in \{1,\dots,r\}$, we have

$$sgn(g_i'(\delta)) = sgn(g_i'(\eta)). \tag{120}$$

Since $\eta \in \overline{F^{(t)}} \cap \overline{G^{(t)}}$, there exist $\delta \in F^{(t)} \cap W$, $\gamma \in G^{(t)} \cap W$ and an $i \in \{1,\dots,r\}$ such that g_i changes sign between δ and γ.

Since x' (resp. x', y') does not change sign between γ and δ this contradicts (120).

Therefore ξ must be special. Let $\delta \in D(\delta_1, \delta_2)$ be the unique point such that

$$x'(\delta) > 0, y'(\delta) = 0.$$

We have $\{\delta\} = D(\delta_1, \delta_2) \cap \{y' = 0\}$. Then

$$D(\delta_1, \delta_2) = \{\delta\} \coprod D(\delta_1, \xi) \coprod D(\xi, \delta_2).$$

Let $\delta_- \in D(\delta_1, \xi)$ be the unique point such that $x'(\delta_-) > 0, y'(\delta_-) < 0$ and $|y'(\delta_-)|^N < |x'(\delta_-)|, \forall N \in \mathbb{N}$. Then $\delta \in \overline{\{\delta_-\}}$, in particular,

$$\delta \in \overline{D(\delta_1, \xi)}. \tag{121}$$

Similarly

$$\delta \in \overline{D(\xi, \delta_2)}. \tag{122}$$

By the previous case, each of $D(\delta_1, \xi)$, $D(\xi, \delta_2)$ is contained either in $F^{(t)}$ or $G^{(t)}$.

Without loss of generality, assume that $D(\delta_1, \xi) \subset F^{(t)}$. By (121) and the relative closedness of $F^{(t)}$, we have $\delta \in F^{(t)}$. By (122) and the relative closedness of $G^{(t)}$, we have $D(\xi, \delta_2) \subset F^{(t)}$, so $D(\delta_1, \delta_2) \subset F^{(t)}$ as desired. □

Corollary 3.5.5 *Let $[\delta_1, \delta_2]$ be a maximal interval. Then $D(\delta_1, \delta_2)$ is entirely contained either in $F^{(t)}$ or in $G^{(t)}$.*

Proof : This follows from the preceding lemma by induction on the number of special points inside $[\delta_1, \delta_2]$.

In order to address the global connectedness, we need a notion of **signed dual graph** associated to a sequence of point blowings up of a point $\epsilon \in \text{Sper}\, A$ and a subset W of Sper A. For each maximal interval I (see Definition 3.2.16), take Sper $A_{jt} \subset X_t$ such that $I \setminus \text{Sper}\ A_{jt}$ is either empty or consists of one distinguished point (such an A_{jt} exists by Proposition 3.2.18). When necessary, we will denote this j by $j(I)$. Let $x_{jt}, y_{jt} \in A_{jt}$ be the elements given in the Definition 3.2.5. By Proposition 3.2.18, we have $I \cap \text{Sper}\ A_{jt} \subset \{x_{jt} = 0\}$.

Let $W^{(t)} = \rho_t^{-1}(W)$. Let I a maximal interval, denote by I° its interior, and $s \in \{+, -\}$, let

$$W(I, s) = \{\delta \in \text{Sper}\, A_{jt} \mid sgn(x_{jt}(\delta)) = s,\ \overline{\{\delta\}} \cap I^\circ \neq \emptyset\}. \tag{123}$$

Definition 3.5.6 *Consider a pair (I, s) as above. We say that (I, s) is admissible if*

$$W^{(t)} \cap \mathit{Sper}\ A_{jt} \supset W(I, s) \neq \emptyset. \tag{124}$$

Consider two admissible pairs (I, s), $(\tilde{I}, \tilde{s})$. We say that these two pairs are **equivalent** if the following conditions hold :

(a) $I \cap \text{Sper}\ A_{jt} \cap \text{Sper}\ A_{\tilde{j}t} = \tilde{I} \cap \text{Sper}\ A_{jt} \cap \text{Sper}\ A_{\tilde{j}t}$,

(b) the sets

$$\{\delta \in \text{Sper}\ A_{jt} \cap \text{Sper}\ A_{\tilde{j}t} \mid sgn(x_{jt}(\delta)) = s\}$$

and

$$\{\delta \in \text{Sper}\ A_{jt} \cap \text{Sper}\ A_{\tilde{j}t} \mid sgn(x_{\tilde{j}t}(\delta)) = \tilde{s}\}$$

coincide in a neighbourhood of $I \cap \text{Sper}\ A_{jt} \cap \text{Sper}\ A_{\tilde{j}t}$.

Given two equivalent admissible pairs (I, s) and $(\tilde{I}, \tilde{s})$, the set of endpoints of I coincides with the set of endpoints of $\tilde{I}$ (viewed as points of the marked real geometric surface X_t). In this way, given an equivalence class of admissible pairs $\{(I, s)\}$, it makes sense to talk about endpoints of $\{(I, s)\}$.

Definition 3.5.7 *1. A vertex of the signed dual graph Γ_t associated to X_t and W is an equivalence class of an admissible pair (I, s), which we will still denote, by abuse of notation, by (I, s).*

2. By definition, two distinct vertices (I, s) and $(\tilde{I}, \tilde{s})$ of Γ_t are connected by an edge of Γ_t if the following conditions hold :

(a) I and $\tilde{I}$ share a common endpoint ξ and suppose that $\tilde{I} \not\subset \{x_{ji} = 0\}$;

(b) we have

$$W^{(t)} \cap \{\delta \in X_t \mid sgn(x_{jt}(\delta)) = s,\ sgn(x_{\tilde{j}t}(\delta)) = \tilde{s},\ \overline{\{\delta\}} \ni \xi\} \neq \emptyset.$$

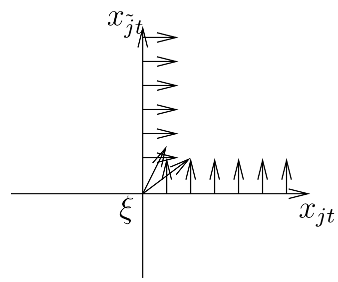

Figure 2: This figure represents an edge of Γ_t connecting two vertices (I, s) and $(\tilde{I}, \tilde{s})$. Here $I = [0, \infty]$, $\tilde{I} = [0, \infty]$, $s = \tilde{s} = +$.

Example: If $W = U$ or $W = V$ then Γ_1 consists of one vertex and no edges (see Fig. (3) for a picture of $U^{(1)}$; the case of V^1 is similar but easier).

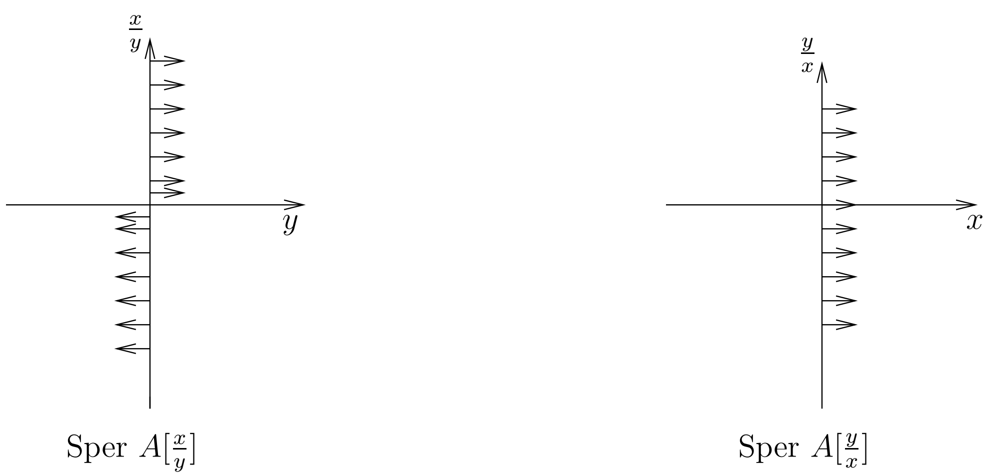

Figure 3: This figure shows the set $U^{(1)}$ in the affine charts

Proposition 3.5.8 *If $W = U$ or $W = V$, the graph Γ_i is a bamboo, that is, a connected, simply connected graph every one of whose vertices belongs to at most two edges.*

Proof: By induction on i. For $i = 1$, the graph consisting of one vertex is connected and satisfies the conclusion of the Proposition. The induction step follows from the next Lemma, which describes the transformation law from Γ_i to Γ_{i+1} in the case when $W = U$ or $W = V$.

Consider the point blowing up $\pi_i : X_{i+1} \to X_i$. Let ξ be the center of the blowing up; recall that, by definition of blowing up in the category of real marked geometric surfaces, ξ belongs to the distinguished set of X_i. Let $\operatorname{Sper} A_{ji}$ be an affine chart of X_i containing ξ. Let $\mathfrak{p}_\xi$ be the support of ξ in A_{ji}. Let k_{ji} be the field of Definition 3.2.5 (2). Let $E_1, \dots, E_p$ be the components of the set $\{x_{ji} = 0\} \cap \rho_i^{-1}(\epsilon)$. Picking a component $E_q,\ q \in \{1, \dots, p\}$ amounts to fixing a total order on k_{ji}, which induces the order on k given by ϵ. For $q \in \{1, \dots, p\}$, let $\{\xi_1^{(q)}, \dots, \xi_\ell^{(q)}\}$ be the set of points of E_q supported at $\mathfrak{p}_\xi$. For each $q \in \{1, \dots, p\}$, the total order on k_{ji} corresponding to E_q induces a total order on the set $\{\xi_1^{(q)}, \dots, \xi_\ell^{(q)}\}$. Renumbering $\{\xi_1^{(q)}, \dots, \xi_\ell^{(q)}\}$, we may assume $\xi_1^{(q)} < \xi_2^{(q)} < \dots < \xi_\ell^{(q)}$.

It follows from the definition of distinguished that one of the points $\xi_t^{(q)}$ is j-distinguished if and only if all of them are.

Fix a pair (q, t), $q \in \{1, \dots, p\}$, $t \in \{1, \dots, \ell\}$. Two cases are possible :

• Case 1 : There exist $a = (I, s),\ b = (\tilde{I}, \tilde{s})$ two vertices of Γ_i connected by an edge (a, b) such that $\xi_t^{(q)}$ is the point common to I and $\tilde{I}$ (note that the pair a, b is not, in general uniquely determined by $\xi_t^{(q)}$). In particular, the points $\xi_t^{(q)}$ are j-distinguished. In this case, we have $p = 1$, so we may denote our points by $\xi_1, \dots, \xi_\ell$. Let $x_{ji}, \tilde{x}_{ji}$ be a privileged regular system of parameters at the points ξ_t.

• Case 2 : We are not in Case 1.

- Case 2.1. : None of the points $\xi_t^{(q)}$ is j-distinguished. Let (x_{ji}, y') be a regular system of parameters of the local ring $A_{\mathfrak{p}_\xi}$. The set $\pi_i^{-1}(\xi_t^{(q)})$ is covered by two affine charts : $\operatorname{Sper} A_{ji}[\frac{x_{ji}}{y'}]$ and $\operatorname{Sper} A_{ji}[\frac{y'}{x_{ji}}]$. Let $x'_{ji} = \frac{x_{ji}}{y'}$.

- Case 2.2 : The point ξ is j-distinguished and lies on the strict transform of $\{x = 0\}$ or $\{y = 0\}$. In this case, $p = \ell = 1$.

Next, we study the neighbourhood of $\pi_i^{-1}(\xi_t^{(q)})$ for each $q \in \{1, \dots, p\},\ t \in \{1, \dots, \ell\}$ and analyze the changes from Γ_i to Γ_{i+1} induced by the blowing-up π_i locally on the part of Γ_i which represents a neighbourhood of $\xi_t^{(q)}$. Since π_i induces an isomorphism outside the points $\xi_t^{(a)}$, the rest of the graph Γ_i remains unchanged under the blowing-up π_i.

In the statement of the following lemma, we refer to the cases 1 and 2 defined above.

Lemma 3.5.9 • *Case 1 : Fix $t \in \{1, \dots, \ell\}$. For each pair of vertices a, b as above, remove the edge (a, b) and add a new vertex c and two new edges (a, c) and (b, c). The graph Γ_{i+1} is obtained from Γ_i by successively performing the above operation for each of $\xi_1, \dots, \xi_\ell$.*

• *Case 2 : Consider a vertex $a = (I, s)$ such that $\xi_t^{(q)} \in I$ for some $t \in \{1, \dots, \ell\}$ and $q \in \{1, \dots, p\}$. Write $I = [\delta_1, \delta_2]$ (again, the vertex a is not, in general, uniquely determined by $\xi_t^{(q)}$).*

- Case 2.1: Take $\lambda \in \{0, \dots, \ell - 1\}$ and $\omega \in \{1, \dots, \ell - \lambda\}$ such that

$$[\delta_1, \delta_2] \cap \{\xi_t^{(q)} \mid t \in \{1, \dots, \ell\},\ q \in \{1, \dots, p\}\} = \{\xi_{\lambda+1}^{(q)}, \xi_{\lambda+2}^{(q)}, \dots, \xi_{\lambda+\omega}^{(q)}\} \tag{125}$$

for some $q \in \{1, \dots, p\}$. Replace a by a bamboo with $2\omega + 1$ vertices. More precisely, we distinguish three cases :

(a) If a belongs to two edges $(a, b),\ (a, c)$ of Γ_i, remove a and the two edges (a, b), (a, c). Introduce the bamboo

$$b \bullet\!-\!\underbrace{\bullet - \bullet - \bullet - \bullet \cdots\cdots \bullet - \bullet}_{2\omega+1} - \bullet\, c$$

(b) If a belongs to only one edge (a, b), remove a and the edge (a, b) and introduce the bamboo

$$b \bullet\!-\!\underbrace{\bullet - \bullet - \bullet - \bullet \cdots\cdots \bullet - \bullet}_{2\omega+1}$$

(c) If a belongs to no edges (in other words, if $i=1$) then

$$\Gamma_2 = \underbrace{\bullet\!-\!\bullet\ -\!\bullet\ -\ \bullet\cdots\cdots\bullet\!-\!\bullet}_{2\omega+1}$$

is a chain of $2\omega+1$ vertices and 2ω edges.

The graph Γ_{i+1} is obtained from Γ_i by performing successively the above operation for each vertex a as above.

- Case 2.2:

(a) $i=1$ and $W=U$, then $\Gamma_2 = \bullet\!-\!\bullet\!-\!\bullet$ is a chain of three vertices and two edges

(b) $i>1$ or $W=V$, then each vertex $a=(I,s)$ such that $\xi\in I$ is an endpoint of Γ_i. For each such vertex a, we add a new vertex b and a new edge (a,b).

Proof: Case 1 : Let $\delta_1,\ \delta_2$ be points of Sper A_{ji} such that $I=[\delta_1,\xi_t]$, $\tilde I=[\xi_t,\delta_2]$. Let $\delta_1'=\pi_i^{-1}(\delta_1)$, $\delta_2'=\pi_i^{-1}(\delta_2)$. Let $x_{ji},x_{\tilde j,i}\in A_{ji}$ be as in the Definition 3.5.6 applied to (I,s) and $(\tilde I,\tilde s)$, respectively. The pair $(x_{ji},x_{\tilde j,i})$ forms a regular system of parameters at ξ_t. Let $x'_{ji}=\frac{x_{ji}}{x_{\tilde j,i}}$ and $x'_{\tilde j,i}=\frac{x_{\tilde j,i}}{x_{ji}}$.

Let

$$\xi_a\in\left\{x'_{ji}=0\right\}\cap\pi_i^{-1}(\xi_t)\subset \text{Sper}\,A_{ji}[x'_{ji}]$$

and

$$\xi_b\in\left\{x'_{\tilde j i}=0\right\}\cap\pi_i^{-1}(\xi_t)\subset \text{Sper}\,A_{ji}[x'_{\tilde j,i}];$$

note that these conditions characterize ξ_a and ξ_b uniquely. Let $J=[\xi_a,\xi_b]$, viewed as a maximal interval of Sper $A_{ji}[x'_{ji}]$. Let $\sigma=s\cdot\tilde s$.

Let a_{i+1},b_{i+1},c_{i+1} be the vertices of Γ_{i+1} defined by $a_{i+1}=([\xi_a,\delta_1'],\sigma)$, $b_{i+1}=([\xi_b,\delta_2'],\sigma)$, $c_{i+1}=(J,\tilde s)$. We have to verify that those three pairs are admissible; first, we will show the admissibility of $([\xi_a,\delta_1'],\sigma)$.

Since (I,s) is admissible, we know that

$$\emptyset\neq W(I,s)\subset W^{(i)}\cap \text{Sper}\,A_{ji}$$

and we need to show that

$$\emptyset\neq W([\xi_a,\delta_1'],\sigma)\subset W^{(i+1)}\cap\text{Sper}\,A_{ji}[x'_{ji}]. \tag{126}$$

To prove (126), note that π_i induces an isomorphism outside the set $\{\xi_{t'}\mid 1\le t'\le\ell\}$; in particular, it induces an isomorphism of a neighbourhood of the open interval (ξ_a,δ_1') onto a neighbourhood of (ξ_t,δ_1). Moreover, the fact that a and b are connected by an edge of Γ_i implies that $sgn(x_{\tilde j,i}(\delta))=\tilde s$ for $\delta\in W(I,s)$. Hence $\delta'\in\pi_i^{-1}(W(I,s))$ if and only if $\overline{\{\delta'\}}\cap(\xi_a,\delta_1')\neq\emptyset$ and $sgn(x'_{ji}(\delta'))=s\cdot\tilde s$. In other words, $W([\xi_a,\delta_1'],\sigma)=\pi_i^{-1}(W(I,s))$. This proves (126), so $([\xi_a,\delta_1'],\sigma)$ is admissible. By symmetry, the pair $([\xi_b,\delta_2'],\sigma)$ is also admissible.

To prove the admissibility of $(J,\tilde s)$, we note that $x_{\tilde j,i}=0$ is the local equation of the exceptional divisor in Sper $A_{ji}[x'_{ji}]$ and hence

$$\pi_i(W(J,\tilde s))=\{\delta\in\text{Sper}\,A_{ji}\mid \overline{\{\delta\}}\ni\xi_t,\ sgn(x_{\tilde j,i}(\delta))=\tilde s,\ \ sgn(x_{ji}(\delta))=s\}.$$

Now the fact that a and b are connected by an edge of Γ_i (see Definition 3.5.7 (b)) implies that

$$\emptyset\neq W(J,\tilde s)\subset W^{(i+1)}\cap\text{Sper}\,A_{ji}[x'_{ji}],$$

so $(J,\tilde s)$ is admissible.

To check that a_{i+1} and c_{i+1} are connected by an edge of Γ_{i+1}, consider the set

$$\{\delta'\in\text{Sper}\,A_{ji}[x'_{ji}]\mid\overline{\{\delta'\}}\ni\xi_a,\ sgn(x_{\tilde j,i}(\delta'))=\tilde s,\ sgn(x'_{ji}(\delta'))=\sigma\}.$$

We have [1]

$$\pi_i(\{\delta' \in \text{Sper}\, A_{ji}[x'_{ji}] \mid \overline{\{\delta'\}} \ni \xi_a,\ sgn(x_{\tilde{j},i}(\delta')) = \tilde{s},\ sgn(x'_{ji}(\delta')) = \sigma\})$$
$$= \{\delta \in \text{Sper}\, A_{ji} \mid \overline{\{\delta\}} \ni \xi_t,\ sgn(x_{\tilde{j},i}(\delta)) = \tilde{s},\ sgn(x_{ji}(\delta)) = s,\ \delta \text{ tangent to } \{x_{ji} = 0\}\}$$
$$\subset \{\delta \in \text{Sper}\, A_{ji} \mid \overline{\{\delta\}} \ni \xi_t,\ sgn(x_{\tilde{j},i}(\delta)) = \tilde{s},\ sgn(x_{ji}(\delta)) = s\} \subset W^{(i)},$$

where the last inclusion comes from the fact that a and b are connected by an edge in Γ_i.

Hence

$$\emptyset \neq \{\delta' \in \text{Sper}\, A_{ji}[x'_{ji}] \mid \overline{\{\delta'\}} \ni \xi_a,\ sgn(x_{\tilde{j},i}(\delta')) = \tilde{s},\ sgn(x'_{ji}(\delta')) = \sigma\}$$
$$\subset W^{(i+1)} \cap \text{Sper}\, A_{ji}[x'_{ji}], \quad (127)$$

which proves that a_{i+1} is connected to c_{i+1}. By symmetry, b_{i+1} is also connected to c_{i+1}.

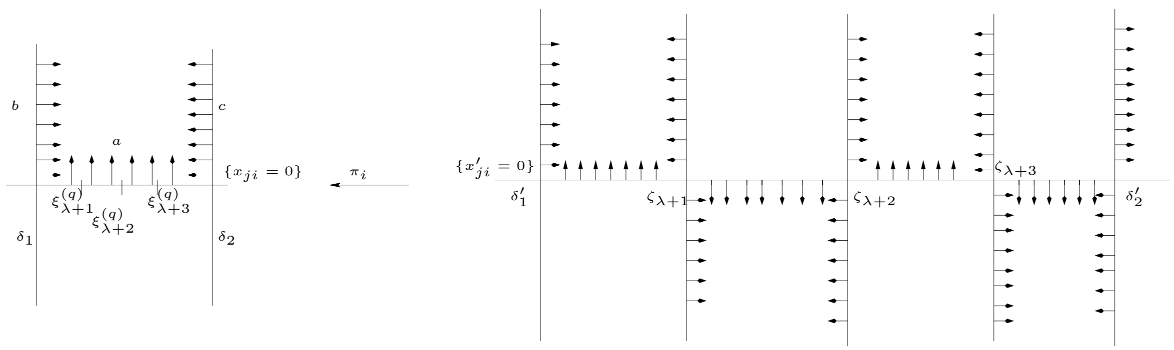

Figure 4: This figure shows, in the Case 2.1 (a), with $\omega = 3$, the transformation of the dual graph under the blowing up π_i.

Case 2.1 (a) Recall that (x_{ji}, y') is the chosen regular system of parameters at $\mathfrak{p}_\xi$.

Let $\delta'_\tau = \pi_i^{-1}(\delta_\tau)$, $\tau \in \{1,2\}$. Let $\zeta_t = \pi_i^{-1}(\xi_t^{(q)}) \cap \{x'_{ji} = 0\}$, $t \in \{\ \lambda+1, \ldots, \lambda+\omega\}$. The new distinguished points in the open interval (δ'_1, δ'_2) are ζ_t, $t \in \{\ \lambda+1, \ldots, \lambda+\omega\}$. The components of $\pi_i^{-1}(\mathfrak{p}_\xi)$ are $\pi_i^{-1}(\xi^{(q)}_{\lambda+1}), \ldots, \pi_i^{-1}(\xi^{(q)}_{\lambda+\omega})$. For $t \in \{\lambda+1, \ldots, \lambda+\omega\}$, let us denote the interval $[-\infty, +\infty] \subset \pi_i^{-1}(\xi_t^{(q)}) \cap \text{Sper}\ A_{ji}[\frac{y'}{x_{ji}}]$ by $[-\infty, \infty]_t$.

Now, there are $2\omega+1$ maximal intervals in $\pi_i^{-1}((\delta_1, \delta_2))$. They are : $[\delta'_1, \zeta_{\lambda+1}]$, $[\zeta_{\lambda+\omega}, \delta'_2]$, $[\zeta_t, \zeta_{t+1}]$, $t \in \{\lambda+1, \ldots, \lambda+\omega-1\}$ and $[-\infty, \infty]_t$, $t \in \{\lambda+1, \ldots, \lambda+\omega\}$.

Without loss of generality, we may assume that $y'(\delta_1) > 0$. Each of this maximal intervals gives rise to an admissible pair as follows.

The intervals $[\zeta_t, \zeta_{t+1}] \subset \{x'_{ji} = 0\}$ give rise to admissible pairs $([\zeta_t, \zeta_{t+1}], (-1)^t \cdot s)$.

We have admissible pairs $([\delta'_1, \zeta_{\lambda+1}], s)$ and $([\zeta_{\lambda+\omega}, \delta'_2], (-1)^\omega \cdot s)$. Finally, the intervals $[-\infty, \infty]_t$ give rise to admissible pairs $([-\infty, \infty]_t, s)$.

To see that the pair $([\zeta_t, \zeta_{t+1}], (-1)^t \cdot s)$ is admissible, we use the fact that π_i is an isomorphism from a neighbourhood of the open interval (ζ_t, ζ_{t+1}) to a neighbourhood of the open interval $(\xi_t^{(q)}, \xi_{t+1}^{(q)})$. Since $y'(\delta_1) > 0$ and since y' changes sign once at each point $\xi_t^{(q)}$ the sign of y' on $(\xi_t^{(q)}, \xi_{t+1}^{(q)})$ is $(-1)^t$. Hence

$$\pi_i(\{\delta' \in X_{i+1} \mid \overline{\{\delta'\}} \cap (\zeta_t, \zeta_{t+1}) \neq \emptyset,\ sgn(x'_{ji}(\delta')) = (-1)^t \cdot s\})$$
$$= \{\delta \in X_i \mid \overline{\{\delta\}} \cap (\xi_t^{(q)}, \xi_{t+1}^{(q)}) \neq \emptyset,\ sgn(x_{ji}(\delta)) = s\}. \quad (128)$$

[1] By δ tangent to $\{x_{ji} = 0\}$, we mean δ such that $\forall N \in \mathbb{N},\ N|x_{ji}(\delta)| < |x_{\tilde{j},i}(\delta)|$

This proves the admissibility of $([\zeta_t, \zeta_{t+1}], (-1)^t \cdot s)$. The proof that $([\delta_1', \zeta_{\lambda+1}], s)$ and $([\zeta_{\lambda+\omega}, \delta_2'], (-1)^\omega \cdot s)$ are admissible is similar and we omit it.

To prove the admissibility of $([-\infty, \infty]_t, s)$, note that

$$\pi_i^{-1}(\{\delta \in X_i \mid \overline{\{\delta\}} \ni \xi_t^{(q)},\ sgn(x_{ji}(\delta)) = s\}) \supset W([-\infty, \infty]_t, s),$$

where the notation $W([-\infty, \infty]_t, s)$ is applied to the affine chart Sper $A_{ji}[\frac{y'}{x_{ji}}]$ and the element $x_{ji} \in$ Sper $A_{ji}[\frac{y'}{x_{ji}}]$.

We claim that the graph Γ_{i+1} contains a bamboo consisting of the above $2\omega + 1$ vertices, arranged in the following order :

$$([\delta_1', \zeta_{\lambda+1}], s),\ ([-\infty, \infty]_{\lambda+1}, s),\ ([\zeta_{\lambda+1}, \zeta_{\lambda+2}], -s),\ ([-\infty, \infty]_{\lambda+2}, s), \\ ([\zeta_{\lambda+2}, \zeta_{\lambda+3}], s), \dots, ([-\infty, \infty]_{\lambda+\omega}, s),\ ([\zeta_{\lambda+\omega}, \delta_2'], (-1)^\omega s). \quad (129)$$

We discuss a sample of edge of this bamboo, for example, $([\zeta_t, \zeta_{t+1}], (-1)^t s),\ ([-\infty, \infty]_{t+1}, s)$. The existence of the other edges can be proved in a similar way.

The two maximal intervals $([\zeta_t, \zeta_{t+1}]$ and $[-\infty, \infty]_{t+1}$ have a common endpoint, namely, ζ_{t+1}. We must show that

$$W^{(i+1)} \cap \{\delta' \in X_{i+1} \mid sgn(x_{ji}'(\delta')) = (-1)^t s,\ sgn(x_{ji}(\delta')) = s,\ \overline{\{\delta'\}} \ni \zeta_{t+1}\} \neq \emptyset.$$

The image of this set under π_i is

$$W^{(i)} \cap \{\delta \in X_i \mid sgn(x_{ji}(\delta)) = s,\ \overline{\{\delta\}} \ni \xi_{t+1}^{(q)},\ \delta \text{ tangent to } \{x_{ji} = 0\}\}$$

and the result follows.

This proves 2.1(a). The cases 2.1(b) and (c) are similar but easier.

Case 2.2: (a) Let $a = (I, s)$. Then $x_{ji} = y$. Put $y' = \frac{x}{y}$; (x_{ji}, y') is a regular system of parameters at ξ. Let $A_{11} = A[x_{ji}, y']$. The point $\xi \in$ Sper A_{11} is the unique point such that supp(ξ)= (x_{ji}, y') and which induces the given order on k. Let $A_{12} = A_{11}[x_{ji}', y']$ where $x_{ji}' = \frac{x_{ji}}{y'}$.

Let $I' \subset \{x_{ji}' = 0\}$ be the 1-maximal interval given by $-\infty \leq y' \leq +\infty$ and $\tilde{I}' \subset \{y' = 0\}$ the 1-maximal interval given by $0 \leq x_{ji}' \leq +\infty$.

Now the vertices of Γ_2 are $(\tilde{I}', +),\ (I', +),\ (\tilde{I}', -)$ with the edges clearly defined.

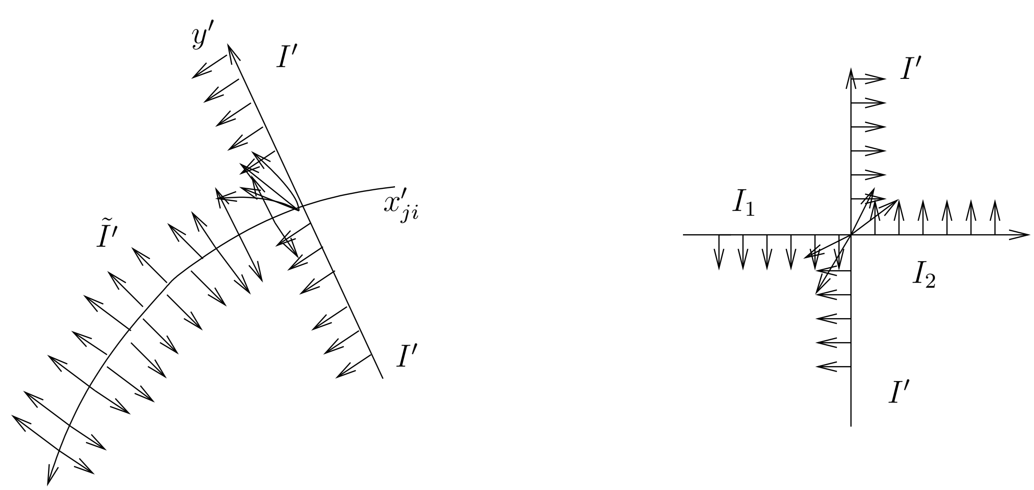

Figure 5: This figure shows the set $U^{(2)}$ in the cases 2.2.a and 2.1.c respectively

(b) Let $a = (I, s)$ be a vertex such that $\xi \in I$; the vertex a is an endpoint of Γ_i. Suppose that $\xi \in \text{Sper}\, A_{ji}$. Let (x_{ji}, y') be a regular system of parameters at ξ. Let $A_{j,i+1} = A_{ji}[x'_{ji}, y']$ where $x'_{ji} = \frac{x_{ji}}{y'}$. Without loss of generality, assume that $x_{ji} > 0, y' > 0$ on $W^{(i)}$.

Let $I' \subset \{x'_{ji} = 0\}$ be the strict transform of I in Sper $A_{j,i+1}$. Then I' is an $(i+1)$-maximal interval. Let $\tilde{I}' \subset \{y' = 0\}$ be the $(i+1)$-maximal interval given by $0 \leq x'_{ji} \leq +\infty$. Now the new vertex b added to Γ_{i+1} is $(\tilde{I}', +)$. It is connected by an edge to a which is represented in Sper $A_{j,i+}$ by $(I', +)$. This completes the proof of Lemma 3.5.9 and with it Proposition 3.5.8. □

Let us finish the proof of Theorem 3.5.1. To each vertex $(I = [\delta_1, \delta_2], s)$ of Γ_t we associate the set $D(\delta_1, \delta_2) \subset U^{(t)}$ which by Corollary 3.5.5 is entirely contained in $F^{(t)}$ or $G^{(t)}$. This defines a partition $\Gamma_F = \left\{(I, s) \;\middle|\; D(\delta_1, \delta_2) \subset F^{(t)}\right\}$, $\Gamma_G = \left\{(I, s) \;\middle|\; D(\delta_1, \delta_2) \subset G^{(t)}\right\}$ of the set of vertices of Γ_t. Assume that $\Gamma_F \neq \emptyset$ and $\Gamma_G \neq \emptyset$. Since Γ_t is connected, there exist $a = ([\delta_{1a}, \delta_{2a}], s_a) \in \Gamma_F$, $b = ([\delta_{1b}, \delta_{2b}], s_b) \in \Gamma_G$ such that (a, b) is an edge of Γ_t. Then $D(\delta_{1a}, \delta_{2a}) \subset F^{(t)}$, $D(\delta_{1b}, \delta_{2b}) \subset G^{(t)}$ and $D(\delta_{1a}, \delta_{2a}) \cap D(\delta_{1b}, \delta_{2b}) \neq \emptyset$. This is a contradiction. This concludes the proof of Theorem 3.5.1. □

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