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Abstract

This paper represents a step in our program towards the proof of the Pierce–Birkhoff conjecture. In the nineteen eighties J. Madden proved that the Pierce-Birkhoff conjecture for a ring A is equivalent to a statement about an arbitrary pair of points $\alpha, \beta \in \operatorname{Sper} A$ and their separating ideal $< \alpha, \beta >$; we refer to this statement as the **local Pierce-Birkhoff conjecture** at α, β. In [21] we introduced a slightly stronger conjecture, also stated for a pair of points $\alpha, \beta \in \operatorname{Sper} A$ and the separating ideal $< \alpha, \beta >$, called the **Connectedness conjecture**. In this paper, for each pair (α, β) with $ht(< \alpha, \beta >) = \dim A$, we define a natural number, called complexity of (α, β). Complexity 0 corresponds to the case when one of the points α, β is monomial; this case was settled in all dimensions in [21]. In the present paper we introduce a new conjecture, called the **Strong Connectedness conjecture**, and prove that the strong connectedness conjecture in dimension $n-1$ implies the connectedness conjecture in dimension n in the case when $ht(< \alpha, \beta >) \leq n-1$. We prove the Strong Connectedness conjecture in dimension 2, which gives the Connectedness and the Pierce–Birkhoff conjectures in any dimension in the case when $ht(< \alpha, \beta >) \leq 2$. Finally, we prove the Connectedness (and hence also the Pierce–Birkhoff) conjecture in the case when $\dim A = ht(< \alpha, \beta >) = 3$, the pair (α, β) is of complexity 1 and A is excellent with residue field $\mathbb{R}$.

1 Introduction

All the rings in this paper will be commutative with 1. Let R be a real closed field. Let $B = R[x_1, \ldots, x_n]$. If A is a ring and $\mathfrak{p}$ a prime ideal of A, $\kappa(\mathfrak{p})$ will denote the residue field of $\mathfrak{p}$.

The Pierce–Birkhoff conjecture asserts that any piecewise-polynomial function $f : R^n \to R$ can be expressed as a maximum of minima of a finite family of polynomials in n variables. We start by giving the precise statement of the conjecture as it was first stated by M. Henriksen and J. Isbell in the early nineteen sixties.

Definition 1.1 *A function $f : R^n \to R$ is said to be* ***piecewise polynomial*** *if R^n can be covered by a finite collection of closed semi-algebraic sets P_i such that for each i there exists a polynomial $f_i \in B$ satisfying $f|_{P_i} = f_i|_{P_i}$.*

Clearly, any piecewise polynomial function is continuous. Piecewise polynomial functions form a ring, containing B, which is denoted by $PW(B)$.

On the other hand, one can consider the (lattice-ordered) ring of all the functions obtained from B by iterating the operations of sup and inf. Since applying the operations of sup and inf to polynomials produces functions which are piecewise polynomial, this ring is contained in $PW(B)$ (the latter ring is closed under sup and inf). It is natural to ask whether the two rings coincide. The precise statement of the conjecture is:

Conjecture 1 (Pierce-Birkhoff) *If $f : R^n \to R$ is in $PW(B)$, then there exists a finite family of polynomials $g_{ij} \in B$ such that $f = \sup_i \inf_j (g_{ij})$ (in other words, for all $x \in R^n$, $f(x) = \sup_i \inf_j (g_{ij}(x))$).*

This paper is a step in a program for proving the Pierce–Birkhoff conjecture. The starting point of this program is the abstract formulation of the conjecture in terms of the real spectrum of B and separating ideals proposed by J. Madden in 1989 [26].

For more information about the real spectrum, see [7]; there is also a brief introduction to the real spectrum and its relevance to the Pierce–Birkhoff conjecture in the Introduction to [21].

Terminology: If A is an integral domain, the phrase "valuation of A" will mean "a valuation of the field of fractions of A, non-negative on A". Also, we will sometimes commit the following abuse of notation. Given a ring A, a prime ideal $\mathfrak{p} \subset A$, a valuation ν of $\frac{A}{\mathfrak{p}}$ and an element $x \in A$, we will write $\nu(x)$ instead of $\nu(x \mod \mathfrak{p})$, with the usual convention that $\nu(0) = \infty$, which is taken to be greater than any element of the value group.

Recall some notation : For a point $\alpha \in$ Sper A we denote by $\mathfrak{p}_\alpha$ the support of α, by $A[\alpha] = \frac{A}{\mathfrak{p}_\alpha}$ and by $A(\alpha)$ the field of fractions of $A[\alpha]$. We also let ν_α denote the valuation associated to α, Γ_α the value group, R_{ν_α} the valuation ring, k_α its residue field and $\mathrm{gr}_\alpha(A)$ the graded ring associated to the valuation ν_α. For $f \in A$ with $\gamma = \nu_\alpha(f)$, let $in_\alpha f$ denote the natural image of f in $\frac{P_\gamma}{P_{\gamma_+}}$. Finally, if k is any field, we denote by $\overline{k}$ its real closure.

Definition 1.2 *Let*

$$f : Sper\ A \to \coprod_{\alpha \in Sper\ A} A(\alpha)$$

be a map such that, for each $\alpha \in Sper\ A$, $f(\alpha) \in A(\alpha)$. We say that f is piecewise polynomial (denoted $f \in PW(A)$) if there exits a covering of Sper A by a finite family $(S_i)_{i \in I}$ of constructible sets, closed in the spectral topology and a family $(f_i)_{i \in I}$, $f_i \in A$ such that, for each $\alpha \in S_i$, $f(\alpha) = f_i(\alpha)$.

We call f_i a local representative of f at α and denote it by f_α (f_α is not, in general, uniquely determined by f and α; this notation means that one such local representative has been chosen once and for all).

Definition 1.3 *A ring A is a Pierce-Birkhoff ring if, for each $f \in PW(A)$, there exists a finite collection $\{f_{ij}\}_{i,j} \subset A$ such that $f = \sup_i \inf_j f_{ij}$.*

The generalized Pierce-Birkhoff Conjecture says:

Conjecture 2 ***(Pierce-Birkhoff Conjecture for regular rings)*** *Let A be a regular ring. Then A is a Pierce-Birkhoff ring.*

Madden reduced the Pierce–Birkhoff conjecture to a purely local statement about separating ideals and the real spectrum. Namely, he introduced

Definition 1.4 *Let A be a ring. For $\alpha, \beta \in$ Sper A, the **separating ideal** of α and β, denoted by $< \alpha, \beta >$, is the ideal of A generated by all the elements $f \in A$ which change sign between α and β, that is, all the f such that $f(\alpha) \geq 0$ and $f(\beta) \leq 0$.*

Definition 1.5 *A ring A is locally Pierce-Birkhoff at α, β if the following condition holds : let f be a piecewise polynomial function, let $f_\alpha \in A$ be a local representative of f at α and $f_\beta \in A$ a local representative of f at β. Then $f_\alpha - f_\beta \in < \alpha, \beta >$.*

Theorem 1.6 *(Madden) A ring A is Pierce-Birkhoff if and only if it is locally Pierce-Birkhoff for all $\alpha, \beta \in$ Sper A.*

Remark 1.7 *Assume that β is a specialization of α. Then*

(1) $< \alpha, \beta > = \mathfrak{p}_\beta$.

(2) $f_\alpha - f_\beta \in \mathfrak{p}_\beta$. Indeed, we may assume that $f_\alpha \neq f_\beta$, otherwise there is nothing to prove. Since $\beta \in \overline{\{\alpha\}}$, f_α is also a local representative of f at β. Hence $f_\alpha(\beta) - f_\beta(\beta) = 0$, so $f_\alpha - f_\beta \in \mathfrak{p}_\beta$.

Therefore, to prove that a ring A is Pierce-Birkhoff, it is sufficient to verify the Definition 1.5 for all α, β such that neither of α, β is a specialization of the other.

In [21], we introduced

Conjecture 3 (the Connectedness conjecture) *Let A be a regular ring. Let $\alpha, \beta \in$ Sper A and let $g_1, \dots, g_s$ be a finite collection of elements of $A \setminus < \alpha, \beta >$. Then there exists a connected set $C \subset$ Sper A such that $\alpha, \beta \in C$ and $C \cap \{g_i = 0\} = \emptyset$ for $i \in \{1, \dots, s\}$ (in other words, α and β belong to the same connected component of the set Sper $A \setminus \{g_1 \dots g_s = 0\}$).*

In the paper [21], we stated the Connectedness conjecture (in the special case when A is a polynomial ring) and proved that it implies the Pierce–Birkhoff conjecture. The same proof shows that the Connectedness Conjecture implies the Pierce-Birkhoff Conjecture for an arbitrary ring.

Definition 1.8 *A subset C of Sper A is said to be* ***definably connected*** *if it is not a union of two non-empty disjoint constructible subsets, relatively closed for the spectral topology.*

Definition 1.9 ***Definable Connectedness Property*** *Let A be a ring. Let $\alpha, \beta \in$ Sper A. We say that A has the Definable Connectedness Property at α, β if, for any finite collection $g_1, \dots, g_s$ of elements of $A \setminus < \alpha, \beta >$, there exists a definably connected set $C \subset$ Sper A such that $\alpha, \beta \in C$ and $C \cap \{g_i = 0\} = \emptyset$ for $i \in \{1, \dots, s\}$ (in other words, α and β belong to the same definably connected component of the set Sper $A \setminus \{g_1 \dots g_s = 0\}$).*

Conjecture 4 (Definable Connectedness Conjecture) *Let A be a regular ring. Then A satisfies the Definable Connectedness Property at any $\alpha, \beta \in$ Sper A.*

Exactly the same proof which shows that the Connectedness Property implies the Pierce-Birkhoff Conjecture applies verbatim to show that the Definable Connectedness Property implies the Pierce-Birkhoff conjecture for any ring A.

One advantage of the Connectedness conjecture is that it is a statement about A (resp. about polynomials if $A = B$) which makes no mention of piecewise polynomial functions.

The Connectedness Conjecture is local in α and β. The purpose of this paper is to associate to each pair (α, β) with $ht(< \alpha, \beta >) = \dim A$ a natural number, called the complexity of (α, β), and prove the Connectedness Conjecture in the simplest case, according to this hierarchy, which is open : that of dimension 3 and complexity 1.

Definition 1.10 *Let k be an ordered field. A* k***-curvette*** *on Sper A is a morphism of the form*

$$\alpha : A \to k\left[\left[t^\Gamma\right]\right],$$

where Γ is an ordered group. A k***-semi-curvette*** *is a k-curvette α together with a choice of the sign data sgn x_1,..., sgn x_r, where $x_1, ..., x_r$ are elements of A whose t-adic values induce an $\mathbb{F}_2$-basis of $\Gamma / 2\Gamma$.*

We explained in [22] how to associate to a point α of Sper A a $\bar{k}_\alpha$-semi-curvette. Conversely, given an ordered field k, a k-semi-curvette α determines a prime ideal $\mathfrak{p}_\alpha$ (the ideal of all the elements of A which vanish identically on α) and a total ordering on $A/\mathfrak{p}_\alpha$ induced by the ordering of the ring $k\left[\left[t^\Gamma\right]\right]$ of formal power series.

Below, we will often describe points in the real spectrum by specifying the corresponding semi-curvettes.

Let $(A, \mathfrak{m}, R)$ be a regular local ring of dimension n and ν a valuation centered in A; let $\Phi = \nu(A \setminus \{0\})$; Φ is a well-ordered set. For an ordinal λ, let γ_λ be the element of Φ corresponding to λ.

Definition 1.11 *A **system of approximate roots** of ν is a countable well-ordered set $\mathbf{Q} = \{Q_i\}_{i\in\Lambda}$, $Q_i \in A$, minimal in the sense of inclusion, satisfying the following condition : for every ν-ideal I in A, we have*

$$I = \left\{ \prod_j Q_j^{\gamma_j} \;\middle|\; \sum_j \gamma_j \nu(Q_j) \geq \nu(I) \right\} A. \tag{1}$$

By definition, each $Q \in \mathbf{Q}$ comes equipped with additional data, called the expression of Q and denoted by $Ex(Q)$. The expression is a sum of generalized monomials involving approximate roots which precede Q in the given order.

A system of approximate roots of ν up to γ_λ is a well-ordered set of elements of A satisfying (1) only for ν-ideals I such that $\nu(I) < \gamma_\lambda$.

*A finite product of the form $\mathbf{Q}^\eta = \prod\limits_j Q_j^{\eta_j}$ with $\eta_j \in \mathbb{N}$ is called a **generalized monomial**. We order the set of generalized monomials by the lexicographical order of the pairs $(\nu(\mathbf{Q}^\eta), \eta)$ (cf. [22], below Definition 1.4).*

In paragraph 1.2, Theorem 1.7 of [22], we constructed a system of approximate roots up to some γ, Q_i, recursively in i. From now on, we fix this system of approximate roots once and for all.

Let $u_1, \dots, u_n$ be a regular system of parameters of A.

Definition 1.12 *Let $i \in \mathbb{N}$ be a natural number, consider an approximate root $(Q, Ex(Q))$. The notion of Q being of complexity i is defined as follows. We say that Q is an approximate root of complexity 0 if $Q \in \{u_1, \dots, u_n\}$. For $i > 0$, we say that Q is of complexity i if all the approximate roots appearing in $Ex(Q)$ are of complexity at most $i-1$ and at least one approximate root appearing in $Ex(Q)$ is of complexity precisely $i-1$.*

Fix $\alpha, \beta \in$ Sper A and consider the Connectedness conjecture for this pair (α, β). Assume $\sqrt{<\alpha, \beta>} = \mathfrak{m}$. We now define a natural number, called the complexity of (α, β).

Definition 1.13 *The complexity of (α, β) is the smallest natural number i such that every ν_α-ideal containing $<\alpha, \beta>$ is generated by generalized monomials involving approximate roots of complexity at most i.*

In [21], we proved the Connectedness conjecture for polynomial rings of arbitrary dimension over a real closed field and pairs (α, β) of complexity 0. Using Corollary 5.2 below, based on [3], Chapter VII, 8.6, we can extend this result to the case of excellent regular local rings A of arbitrary dimension and pairs (α, β) of complexity 0.

In this paper, we will assume that $R = \mathbb{R}$. In this case, $Ex(Q)$ is a binomial in the approximate roots preceding Q as we show below. The main result of this paper is :

Theorem 1.14 *Let $(A, \mathfrak{m}, \mathbb{R})$ be an excellent 3-dimensional regular local ring such that $\mathbb{R} \hookrightarrow A$. Let $\alpha, \beta \in$ Sper A. Assume that one of the following holds :*

(1) $ht(<\alpha, \beta>) \leq 2$

(2) $ht(<\alpha,\beta>) = 3$ and either $\{u_1,u_2,u_3\} \cap <\alpha,\beta> \neq \emptyset$ or (α,β) is of complexity at most 1.
Then the Connectedness Conjecture (and hence the Local Pierce-Birkhoff Conjecture) holds for (α,β).

Fix $\alpha,\beta \in \text{Sper } A$ and let $\mathfrak{p} = \sqrt{<\alpha,\beta>}$. The case when $ht(\mathfrak{p}) = 1$ is easy. The proof given in [22] works verbatim in any dimension.

The present paper is organized as follows.

In §2 we state a new conjecture, called the Strong Connectedness Conjecture. We show that the Strong connectedness conjecture in dimension $n-1$ implies the Connectedness conjecture in dimension n whenever $ht(\mathfrak{p}) < \dim\ A$.

In §3 we prove the Strong Connectedness Conjecture for arbitrary regular local rings of dimension 2. We deduce the Connectedness Property and the local Pierce-Birkhoff Conjecture for any ring A (of any dimension) and $\alpha,\beta \in \text{Sper } A$ such that $ht(<\alpha,\beta>) = 2$ and $A_{\sqrt{<\alpha,\beta>}}$ is regular.

§4 is devoted to the study of graded algebras associated to points of real spectra in the case when the residue field of our local ring is $\mathbb{R}$.

In §5 we prove a comparison theorem between connected components of a constructible subset $C \subset \text{Sper } A$ and those of the set $\tilde{C} \subset \text{Sper } R[u_1,\dots,u_n]_{(u_1,\dots,u_n)}$ defined by the same formulae as C.

Finally, we describe some subsets of Sper A, containing α and β, which will be later proved to be connected, thus verifying the Connectedness Conjecture.

In §6 we prove the Connectedness conjecture in the Case 2 of the Theorem 1.14.

2 The Strong Connectedness Conjecture

Let $\dim\ A = 3$ and $ht\ \mathfrak{p} = 2$. A natural idea would be to apply the already known 2-dimensional connectedness conjecture to the regular 2-dimensional local ring $A_\mathfrak{p}$. Then one would construct a sequence of point blowings up $\tilde{\pi} : \tilde{X}_{\tilde{l}} \to \text{Sper } A_\mathfrak{p}$ and a connected set in $\tilde{X}_{\tilde{l}}$ satisfying the conclusion of the conjecture. Finally, we would construct a sequence $\pi : X_l \to \text{Sper } A$ of blowings up of points and smooth curves whose restriction to the generic point of $V(\mathfrak{p})$ is $\tilde{\pi}$.

The difficulty with this approach is that the 2-dimensional connectedness conjecture cannot be applied directly. Indeed, let $g_1,\dots,g_s$ be as in the connectedness conjecture and let $\Delta_\alpha \subset \Gamma_\alpha$ denote the greatest isolated subgroup not containing $\nu_\alpha(\mathfrak{p})$.
The hypothesis $g_i \notin <\alpha,\beta>$ does not imply that $g_i \notin <\alpha,\beta> A_\mathfrak{p}$: it may happen that $\nu_\alpha(g_i) < \nu_\alpha(<\alpha,\beta>)$, $\nu_\alpha(g) - \nu_\alpha(\mathfrak{p}) \in \Delta_\alpha$ and so $g_i \in <\alpha,\beta> A_\mathfrak{p}$, as we show by the example below.

Example. Let α,β be given by the curvettes

$$x(t) = t^{(0,3)} \tag{2}$$
$$y(t) = t^{(0,4)} + bt^{(1,0)} \tag{3}$$
$$z(t) = t^{(0,5)} + ct^{(1,1)}, \tag{4}$$

where $b \in \{b_\alpha, b_\beta\} \subset \mathbb{R}$ and $c \in \{c_\alpha, c_\beta\} \subset \mathbb{R}$ and $t^{(0,1)} > 0, t^{(1,0)} > 0$. The constants $b_\alpha, b_\beta, c_\alpha, c_\beta$ will be specified later. Let $f_1 = xz - y^2$, $f_2 = x^3 - yz$, $f_3 = x^2y - z^2$; consider the ideal (f_1,f_2,f_3). The most general common specialization of α,β is given by the curvette

$$x(t) = t^3 \tag{5}$$
$$y(t) = t^4 \tag{6}$$
$$z(t) = t^5, \tag{7}$$

$t > 0$. The corresponding point of Sper A has support (f_1,f_2,f_3), so $\mathfrak{p} = \sqrt{<\alpha,\beta>} = (f_1,f_2,f_3)$. Let $(x_\alpha(t), y_\alpha(t), z_\alpha(t))$ and $(x_\beta(t), y_\beta(t), z_\beta(t))$ be the curvettes defining α and

β as in (2)–(4). Let us calculate $f_1(x_\alpha(t), y_\alpha(t), z_\alpha(t))$ and $f_1(x_\beta(t), y_\beta(t), z_\beta(t))$. In the notation of (2)–(4) we have

$$\begin{aligned} f_1(x(t), y(t), z(t)) &= (c-2b)t^{(1,4)} + \tilde{f}_1 && (8)\\ f_2(x(t), y(t), z(t)) &= -(c+b)t^{(1,5)} + \tilde{f}_2 && (9)\\ f_3(x(t), y(t), z(t)) &= (b-2c)t^{(1,6)} + \tilde{f}_3, && (10)\end{aligned}$$

where $\tilde{f}_i$ stands for higher order terms with respect to the t-adic valuation. Choose $b_\alpha, b_\beta, c_\alpha, c_\beta$ so that none of f_1, f_2, f_3 change sign between α and β. The smallest ν_α value of an element which changes sign between α and β is $(1,4)+(0,4) = (1,5)+(0,3) = (1,8)$, so $\nu_\alpha(<\alpha,\beta>) = (1,8)$. Thus we have $f_i \notin <\alpha,\beta>$, but $f_i \in <\alpha,\beta> A_\mathfrak{p}$, as desired.

Thus we are naturally led to formulate a stronger version of the Connectedness Conjecture, one which has exactly the same conclusion but with somewhat weakened hypotheses. This phenomenon occurs in all dimensions, as we now explain.

Definition 2.1 ***Strong Connectedness Property*** *Let Σ be a ring, $\alpha, \beta \in$ Sper Σ, having a common specialization ξ. We say that Σ has the Strong Connectedness Property at α, β if given any $g_1, \dots, g_s \in \Sigma \setminus (\mathfrak{p}_\alpha \cup \mathfrak{p}_\beta)$ such that for all $j \in \{1, \dots, s\}$,*

$$\nu_\alpha(g_i) \le \nu_\alpha(<\alpha,\beta>), \quad \nu_\beta(g_i) \le \nu_\beta(<\alpha,\beta>) \tag{11}$$

and such that no g_i changes sign between α and β, the points α and β belong to the same connected component of Sper $\Sigma \setminus \{g_1 \cdots g_s = 0\}$.

Conjecture 5 ***Strong Connectedness Conjecture*** *Let Σ be a regular ring. Then Σ has the Strong Connectedness Property at any pair of points $\alpha, \beta \in$ Sper Σ having a common specialization.*

Let A be a ring and $\alpha, \beta \in$ Sper A. Let $\mathfrak{p} = \sqrt{<\alpha,\beta>}$, let α_0 be the pre-image of α under the natural inclusion σ : Sper $A_\mathfrak{p} \hookrightarrow$ Sper A and similarly for β_0.

Theorem 2.2 *If Sper $A_\mathfrak{p}$ has the Strong Connectedness property at α_0, β_0, then A satisfies the Connectedness Conjecture at α, β.*

Proof : Let $g_1, \dots, g_s \in A$ be the elements appearing in the statement of the Connectedness Conjecture. Renumbering the g_i, if necessary, we may assume that $g_1, \dots, g_l \notin <\alpha_0, \beta_0>$ and $g_{l+1}, \dots, g_s \in <\alpha_0, \beta_0>$. The condition $g_{l+1}, \dots, g_s \in <\alpha_0, \beta_0>$ implies that, for $i \in \{l+1, \dots, s\}$, $\nu_{\alpha_0}(g_i) = \nu_{\alpha_0}(<\alpha_0, \beta_0>)$.

By hypothesis, there exists a connected set $C_0 \subset$ Sper $A_\mathfrak{p}$, $\alpha_0, \beta_0 \in C_0$ such that $C_0 \subset \{g_1 \cdots g_s \ne 0\}$. Then $\sigma(C_0)$ satisfies the conclusion of the Connectedness Conjecture for $A, \alpha, \beta, g_1, \dots, g_s$. $\square$

In the next section we will use Zariski's theory of complete ideals to prove the Strong Connectedness Conjecture in dimension 2, and hence also the Connectedness conjecture in dimension 3, when $ht(\mathfrak{p}) = 2$.

3 The case when the height of $\mathfrak{p}$ is 2

Theorem 3.1 *Conjecture 5 is true when Σ is of dimension 2.*

Proof : If one of α, β is a specialization of the other, the result is trivially true, because the connected component of Sper $\Sigma \setminus \{g_1 \cdots g_s = 0\}$ containing the more general point among α and β satisfies the conclusion of the conjecture. From now on we shall assume that none of α and β is a specialization of the other.

Let z be a new variable. We will say that a point $\eta \in$ Sper $k[z]$ is closed if $\{\eta\} = \overline{\{\eta\}}$.

Consider a point $\alpha \in$ Sper Σ, dim $\Sigma = 2$. Let ξ be the most special specialization of α. Assume that $\mathrm{ht}(\mathfrak{p}_\xi) = 2$ and $\alpha \neq \xi$. Let (x, y) be a regular system of parameters of $\Sigma_{\mathfrak{p}_\xi}$ and let k be the residue field $k = \frac{\Sigma}{\mathfrak{p}_\xi}$. Let $\rho : X \to$ Sper Σ be the blowing up of Sper Σ along (x, y). Let α' be the strict transform of α in X (see [22], Definitions 3.19 and 3.20). If $\nu_\alpha(y) \geq \nu_\alpha(x)$ then $\alpha' \in$ Sper $\Sigma[\frac{y}{x}]$. Consider the homomorphism $\Sigma[\frac{y}{x}] \to k[z]$ which maps $\frac{y}{x}$ to z and elements of Σ to their image in k. In this way, we identify Sper $k[z]$ with Sper $\Sigma[\frac{y}{x}] \cap \rho^{-1}(\xi)$.

Definition 3.2 *The slope of α, denoted by $sl(\alpha)$, is the following element of Sper $k[z] \cup \{\infty\}$*

- if $\nu_\alpha(x) > \nu_\alpha(y)$, $sl(\alpha) := \infty$;

- if $\nu_\alpha(x) \leq \nu_\alpha(y)$, $sl(\alpha)$ is the most special specialization of α' in Sper $\Sigma[\frac{y}{x}]$.

Let $\alpha, \beta \in$ Sper Σ be the two points centered at ξ and having the same slope. We say that α and β point in the same direction if $sgn(x(\alpha)) = sgn(x(\beta))$ when $sl(\alpha) \neq \infty$ (resp. $sgn(y(\alpha)) = sgn(y(\beta))$ when $sl(\alpha) = \infty$). Otherwise we say that α and β point in different direction.

Examples : Let $\Sigma = \mathbb{Q}[x, y]$.

1. Let α be the point of Sper Σ given by the following semi-curvette $\mathbb{Q}[x, y] \hookrightarrow \mathbb{Q}(\pi)[[t]]$ such that $x \mapsto t$, $y \mapsto \pi t$. Then ξ is the closed point with support (x, y) and the slope of α is the point of Sper $\mathbb{Q}[z]$ such that for any rational number p/q we have $z > p/q \iff \pi > p/q$.

2. Let α be a point of Sper Σ such that $\nu_\alpha(x) = \nu_\alpha(y) > 0$, $\nu_\alpha(y^2 - 2x^2) > 2\nu_\alpha(x)$. Then ξ is the closed point with support (x, y) and the slope of α is the point of Sper $\mathbb{Q}[z]$ with support $(z^2 - 2)$.

Remark 3.3 *In the situation of Definition 3.2, assume that $sl(\alpha) \neq \infty$. Let $k[z](sl(\alpha))$ be the field of fractions of $\frac{k[z]}{\mathfrak{p}_{sl(\alpha)}}$. We can naturally identify $k[z](sl(\alpha))$ with the ordered sub-field of k_α generated over k by the image of $\frac{y}{x}$. The field $k[z](sl(\alpha))$ is a simple extension of k which can be algebraic as in the Example 2 above, or transcendental as in the Example 1.*

Definition 3.4 *Let $f \in$ Sper Σ. We say that $f = 0$ is tangent to α if $\nu_\alpha(f) > \nu_\alpha(\mathfrak{p}_\xi)$.*

First assume that α and β have the same tangent, and that they are facing in different directions along that tangent. Then $< \alpha, \beta >= \mathfrak{p}_\xi$. We want to show that, for all i, $g_i \notin \mathfrak{p}_\xi$. Assume that $g_i \in \mathfrak{p}_\xi$. Write $g_i = ax + by + \tilde{g}_i$ where $a, b \in \Sigma$ and $\tilde{g}_i \in (x, y)^2$. We may assume that the common slope to α and β is not ∞. Then

$$\begin{aligned} \nu_\alpha(g_i) &= \nu_\alpha(\mathfrak{p}_\xi) = \nu_\alpha(x) \leq \nu_\alpha(y) && (12) \\ \nu_\beta(g_i) &= \nu_\beta(\mathfrak{p}_\xi) = \nu_\beta(x) \leq \nu_\beta(y). && (13) \end{aligned}$$

Hence either $a \notin \mathfrak{p}_\xi$ or $(\nu_\alpha(x) = \nu_\alpha(y))$ and $b \notin \mathfrak{p}_\xi$. In particular, $sgn_\alpha(g_i) = sgn_\alpha(ax + by)$ and similarly for sgn_β.

Let $k[z](sl(\alpha))$ be as in the previous remark. By (12) and (13), the natural image of $a + b\frac{y}{x}$ in $k[z](sl(\alpha))$is non zero. Since α and β have the same slope, they induce the same order on $k[z](sl(\alpha))$. Hence $a + b\frac{y}{x}$ does not change sign between α and β, so $x(a + b\frac{y}{x})$ changes sign between α and β, which is a contradiction. Hence $g_i \notin \mathfrak{p}_\xi$. Then a small connected neighbourhood U (small enough so that $\{g_1 \cdots g_s = 0\} \cap U = \emptyset$) of ξ satisfies the conclusion of Conjecture 5. This proves the Theorem in the special case when α and β have the same slope but point in different directions.

From now on assume that if α and β have the same slope, they point in the same direction.

Let $\pi : X' \to X =$ Sper A the shortest sequence of blowings up such that the strict transforms α' and β' of α and β have the same specialization ξ' with $ht(\mathfrak{p}_{\xi'}) = 2$ and distinct slopes (see [22], by iterating Proposition 3.31). Note that, if g'_i denotes the strict transform of g_i, then the g'_i such that $g'_i(\xi) \neq 0$ play no role and if $g'_i(\xi) = 0$, by (11), $\{g'_i = 0\}$ cannot be tangent to α' or β' or to the last exceptional divisor if it exists. Let $\mathcal{O}_{X',\xi'}$ be the local ring of X' at ξ' and let x', y' be a regular system of parameters such that $\{x'=0\}$ is the last exceptional divisor if it exists and $\{y'=0\}$ the second one if any. In the case we had not to blow up, we take an x' such that $\{x'=0\}$ is not tangent to α', β' or any of $\{g'_i=0\}$ and such that $x'(\alpha')>0$ and $x'(\beta')>0$. Note that x' does not change sign between α and β (otherwise the blowing up sequence π would have stopped at an earlier stage). Replacing x' by $-x'$ if necessary, we may assume that $x'(\alpha')>0$, $x'(\beta')>0$.

Let us introduce the following total ordering on the set $\{g'_1,\dots,g'_s\}$. Write each g'_j as a formal power series in the formal completion $\mathcal{O}_{X',\xi'} \to k'[[x',y']]$ as

$$g'_j = y' + \sum_{i=1}^{\infty} c_{ij} {x'}^i \text{ with } c_{ij} \in k'.$$

This is possible because of the choice of x', y', the non tangency of $\{g'_j=0\}$ with α', β' and the last exceptional divisor. We compare g'_j and g'_ℓ by comparing the monomials in lexicographic ordering. Namely, we take the smallest i such that $c_{ij} \neq c_{i\ell}$ and we say that $j \prec \ell$ if $c_{ij} < c_{i\ell}$. Without loss of generality, we may assume that $g'_1(\alpha')>0,\dots,g'_\ell(\alpha')>0$, $g'_{\ell+1}(\alpha')<0,\dots,g'_s(\alpha')<0$ and also that $1 \prec \ldots \prec \ell$, $\ell+1 \prec \ldots \prec s$.

Lemma 3.5 *Let $j,q \in \{1,\dots,\ell\}$, $j \prec q$. Then $\{g'_j>0, x'>0\} \subset \{g'_q>0, x'>0\}$.*
Let $j,q \in \{\ell+1,\dots,s\}$, $j \prec q$. Then $\{g'_q>0, x'>0\} \subset \{g'_j>0, x'>0\}$.

Proof : In the first case, we have to prove that $g'_q(\delta)>0 \Rightarrow g'_j(\delta)>0$. Write $g'_j = y' + c_{1j}x' + \cdots + c_{ij}{x'}^i + {x'}^{i+1}(\cdots)$ and $g'_q = y' + c_{1q}x' + \cdots + c_{iq}{x'}^i + {x'}^{i+1}(\cdots)$ with $c_{kj}=c_{kq}$ for $k=1,\dots,i-1$ and $c_{ij}<c_{iq}$. We have $g'_q - g'_j = (c_{iq}-c_{ij}){x'}^i u$ where u is a positive unit of $k'[[x',y']]$. So $g'_q - g'_j = d{x'}^i$ in $\mathcal{O}_{X',\xi'}$ with $d \in \mathcal{O}_{X',\xi'} \setminus \mathfrak{m}_{X',\xi'}$ such that $d = c_{iq} - c_{ij} \mod \mathfrak{m}_{X',\xi'}$, in particular $d(\delta)>0$.

And the same with the second inclusion. □

Lemma 3.6 *We have $c_{11} > c_{1s}$.*

Proof : Note that we have

$$g'_1(\alpha') = y'(\alpha') + c_{11}x'(\alpha') + x'(\alpha')^2 h_1 > 0 \tag{14}$$
$$g'_1(\beta') = y'(\beta') + c_{11}x'(\beta') + x'(\beta')^2 h_1 > 0 \tag{15}$$
$$g'_s(\alpha') = y'(\alpha') + c_{1s}x'(\alpha') + x'(\alpha')^2 h_s < 0 \tag{16}$$
$$g'_s(\beta') = y'(\beta') + c_{1s}x'(\beta') + x'(\beta')^2 h_s < 0 \tag{17}$$

where $h_1, h_s \in k'[[x',y']]$.

Write α' as curvette :

$$\begin{aligned} x'(t) &= t^{\nu_\alpha(x')} + \cdots \\ y'(t) &= b_\alpha t^{\nu_\alpha(x')} + \cdots \end{aligned}$$

where b_α is the natural image of $\frac{y'}{x'}$ in k_α.

Then $g'_1(\alpha')>0 \Leftrightarrow y' + c_{11}x' + {x'}^2 h_1 > 0 \Leftrightarrow y'(t) + c_{11}x'(t) + x'(t)^2 h_1(t) = (b_\alpha + c_{11})t^{\nu_\alpha(x')} + \cdots > 0$ in $\overline{k_\alpha}[[t^{\Gamma_\alpha}]]$, so $b_\alpha + c_{11} \geq 0$ in k_α. By the same arguments, we have : $b_\alpha + c_{1s} \leq 0$ in k_α, $b_\beta + c_{11} \geq 0$ and $b_\beta + c_{1s} \leq 0$ in k_β. If we had $b_\alpha + c_{11} = 0$ and $b_\beta + c_{11} = 0$, then $b_\alpha = b_\beta = -c_{11} \in k'$. Hence $k[z](sl(\alpha')) = k[z](sl(\beta'))$, which contradicts the fact that α' and β' have different slopes. Thus, at least one of the inequalities (14) and (15) is strict, say $b_\alpha + c_{11} > 0$ for instance. Together with the inequality (16), this implies that $c_{11} > c_{1s}$. □

Let

$$C' = \{g'_1 > 0, g'_s < 0\}.$$

By definition C' contains α' and β', so is non empty.

Corollary 3.7 *For any $\delta \in C'$, $x'(\delta) > 0$.*

Proof : It's a straightforward consequence of the preceding Lemma and proof. Of course, $g'_1(\delta) > 0 \Rightarrow x'(\delta)(b_\delta + c_{11}) \geq 0$ and $g'_s(\delta) < 0 \Rightarrow x'(\delta)(b_\delta + c_{1s}) \leq 0$ where b_δ is defined in a similar way as b_α or b_β. So $x'(\delta)(c_{11} - c_{1s}) \geq 0$, which proves the result. □

By Lemma 3.5, for any $\delta \in C$, we have $g'_1(\delta) > 0, \dots, g'_\ell(\delta) > 0$, $g'_{\ell+1}(\delta) < 0, \dots, g'_s(\delta) < 0$. So, finally, being a quadrant in Sper $\mathcal{O}_{X',\xi'}$, if $\mathcal{O}_{X',\xi'}$ is an excellent ring, C' is connected by ([22], Theorem 3.35). So the image C of C' in X satisfies the conclusion of the Strong Connectedness Conjecture. □

Corollary 3.8 *Let A be a ring, $\alpha, \beta \in$ Sper A, $\mathfrak{p} = \sqrt{<\alpha, \beta}$. Assume that the local ring $A_\mathfrak{p}$ is excellent regular of dimension at most 2. Then A has the Connectedness Property at α, β and hence satisfies the local Pierce-Birkhoff conjecture at α, β.*

This follows immediately from Theorem 2.2 and Theorem 3.1.

Remark 3.9 *All the results of this section remain true, with the same proofs if we drop the excellence hypothesis on A, but replace the Strong Definable Connectedness Conjecture by Definable Strong Connectedness Conjecture.*

4 Graded algebras in the case of residue field $\mathbb{R}$

Theorem 4.1 *Let $(\Sigma, \mathfrak{m})$ be a local ring with residue field $\mathbb{R}$. Let $\alpha \in$ Sper Σ such that ν_α is centered at $\mathfrak{m}$. For every $\gamma \in \Gamma_\alpha$, we have $\frac{P_\gamma}{P_{\gamma_+}} \cong \mathbb{R}$.*

Proof : We have non-canonical inclusions

$$\mathbb{R} \subset \frac{P_\gamma}{P_{\gamma_+}} \subset k_\alpha.$$

Thus it is sufficient to prove that $\mathbb{R} \equiv k_\alpha$. Take an element $\overline{b} \in k_\alpha$ and let b be a representative of $\overline{b}$ in R_α. By definitions, there exists $a \in \Sigma[\alpha]$ such that $|b| \leq a$. Since ν_α is centered at $\mathfrak{m}$, we may take a to be a unit of $\Sigma[\alpha]$. Let $\overline{a}$ be the image of a in $\frac{\Sigma[\alpha]}{\mathfrak{m}\Sigma[\alpha]} \cong \mathbb{R}$. Then $|\overline{b}| \leq \overline{a}$. Hence $\overline{b} \in \mathbb{R}$ as desired. □

Corollary 4.2 *Assume in addition that Σ is regular and $\mathbb{R} \subset \Sigma$. Let Q be an approximate root for α, then $Ex(Q)$ is a difference of two generalized monomials in the approximate roots preceding Q.*

Proof : By the construction of approximate roots ([22], section 1.2), $Ex(Q)$ comes from a certain $\mathbb{R}$-linear dependence relation among generalized monomials in approximate roots, preceding Q, of the same value. According to Theorem 4.1, any two such monomials are $\mathbb{R}$-multiples of each other. Now the result follows from the construction of approximate roots. □

5 Connectedness properties

Theorem 5.1 *Let $(A, \mathfrak{m}, R)$ be an excellent regular local ring such that $R \subset A$. Let $(u_1, \dots, u_n)$ be a regular system of parameters of A. Let $C \subset$ Sper A be a constructible set such that all the elements of A appearing in formulae defining C belong to $R[u_1, \dots, u_n]_{(u_1,\dots,u_n)}$. Let $\tilde{C} \subset$ Sper $R[u_1, \dots, u_n]_{(u_1,\dots,u_n)}$ denote the constructible set defined by the same formulae as C. Let*

$$U = \{\delta \in \textit{Sper } A \mid \delta \textit{ is centered at } \mathfrak{m}\}$$

$$\tilde{U} = \{\delta \in \textit{Sper } R[u_1, \dots, u_n]_{(u_1,\dots,u_n)} \mid \delta \textit{ is centered at } (u_1, \dots, u_n)\}.$$

Then the natural map Sper $A \to$ Sper $R[u_1,\dots,u_n]_{(u_1,\dots,u_n)}$ induces a bijection between the set of connected components of $C \cap U$ and the set of connected components of $\tilde{C} \cap \tilde{U}$.

Proof : Consider the following natural ring homomorphisms

$$R[u_1,\dots,u_n]_{(u_1,\dots,u_n)} \xrightarrow{\sigma_0} A \xrightarrow{\sigma} R[[u_1,\dots,u_n]]\ .$$

The theorem follows from ([3], chap. VII, Proposition 8.6) applied to the rings A and $R[u_1,\dots,u_n]_{(u_1,\dots,u_n)}$. $\square$

Corollary 5.2 *Let $(A,\mathfrak{m},R)$ be an excellent regular local ring such that $R \subset A$. Let $(u_1,\dots,u_n)$ be a regular system of parameters of A. Fix a subset $J \subset \{1,\dots,n\}$ and the point $\xi \in$ Sper A such that $\mathfrak{p}_\xi = \mathfrak{m}$. Let U denote the subset of Sper A consisting of generizations of ξ. Let C denote the subset of U defined by specifying sgn u_q (which can be either strictly positive on all of C or strictly negative on all of C) for $q \in J$ and by imposing, in addition, finitely many monomial inequalities of the form*

$$\left|d_i \underline{u}^{\lambda_i}\right| \geq \left|\underline{u}^{\theta_i}\right|, \quad 1 \leq i \leq M \tag{18}$$

where $d_i \in R \setminus \{0\}$, $\lambda_i, \theta_i \in \mathbb{N}^n$ and u_q may appear only on the right hand side of the inequalities (18) for $q \not\in J$. Then C is connected.

Proof : Write $\lambda_i = (\lambda_{1i},\dots,\lambda_{ni})$ and similarly for θ_i. It is sufficient to prove that any two elements of C belong to the same connected component of C.

Consider the natural homomorphism

$$A \to \hat{A} = R[[u_1,\dots,u_n]]. \tag{19}$$

Let $\hat{\xi}$ denote the point of Sper $\hat{A}$ with support $\mathfrak{m}\hat{A}$.

Following ([3], chap. VII, proposition 8.6), C is connected if and only if

$$\hat{C} = \{\delta \in \text{Sper } R[[u_1,\dots,u_n]] \mid u_j(\delta) > 0,\ j \in J, \left|d_i \underline{u}^{\lambda_i}\right| \geq \left|\underline{u}^{\theta_i}\right|,\ 1 \leq i \leq M, \hat{\xi} \in \overline{\{\delta\}}\}$$

is connected (this is where we are using the fact that A is excellent). So it suffices to prove that $\hat{C}$ is connected.

By the preceding Theorem, $\hat{C}$ is connected if and only if the set

$$\begin{aligned} C_\dagger \quad = \quad & \{\delta \in \text{Sper } R[u_1,\dots,u_n]_{(u_1,\dots,u_n)} \mid u_j(\delta) > 0,\ j \in J, \\ & \left|d_i \underline{u}^{\lambda_i}\right| \geq \left|\underline{u}^{\theta_i}\right|,\ 1 \leq i \leq M, \delta \text{ is centered at } (u_1,\dots,u_n)\} \end{aligned}$$

is connected.

Define

$$\begin{aligned} C_0 \quad = \quad & \{\delta \in \text{Sper } R[u_1,\dots,u_n] \mid u_j(\delta) > 0,\ j \in J, \\ & \left|d_i \underline{u}^{\lambda_i}\right| \geq \left|\underline{u}^{\theta_i}\right|,\ 1 \leq i \leq M, \delta \text{ is centered at } (u_1,\dots,u_n)\} \end{aligned}$$

and

$$\begin{aligned} C_{loc} \quad = \quad & \{\delta \in \text{Sper } R[u_1,\dots,u_n]_{\prod_{j\in J} u_j} \mid u_j(\delta) > 0,\ j \in J, \\ & \left|d_i \underline{u}^{\lambda_i}\right| \geq \left|\underline{u}^{\theta_i}\right|,\ 1 \leq i \leq M, \delta \text{ is centered at } (u_1,\dots,u_n)\} \end{aligned}$$

The natural maps $\phi : R[u_1,\dots,u_n] \to R[u_1,\dots,u_n]_{(u_1,\dots,u_n)}$ and $\psi : R[u_1,\dots,u_n] \to R[u_1,\dots,u_n]_{\prod_{j\in J} u_j}$ induce homeomorphisms $\phi|_{C_0} : C_0 \cong C_{loc}$ and $\psi|_{C_0} : C_0 \cong C_\dagger$.

So it suffices to prove that C_{loc} is connected. But

$$C_{loc} = \bigcap_{N\in\mathbb{N}} C_N$$

where

$$\begin{aligned} C_N &= \{\delta \in \text{Sper } R[u_1, \dots, u_n]_{\prod_{j\in J} u_j} \mid \frac{1}{N} \geq u_j(\delta) \geq 0,\ j \in J, \\ &\quad \left| d_i \underline{u}^{\lambda_i} \right| \geq \left| \underline{u}^{\theta_i} \right|,\ 1 \leq i \leq M\}. \end{aligned}$$

By Lemma 4.1 of [21], each C_N is a non empty closed connected subset of Sper $R[u_1, \dots, u_n]_{\prod_{j\in J} u_j}$, hence C_{loc} is connected by ([21], lemma 7.1). $\Box$

Remark 5.3 *Keep the hypothesis of Corollary 5.2. Consider a set $\tilde{C}$ defined by inequalities*

$$\left| \tilde{d}_i \underline{u}^{\lambda_i} \right| \geq \left| \underline{u}^{\theta_i} \right|,\ 1 \leq i \leq M,\ \tilde{d}_i \in A \setminus \mathfrak{m} \tag{20}$$

and the same sign conditions as C. For each i, $1 \leq i \leq M$, take $d'_i \in R$ such that $\left| \tilde{d}_i(\xi) \right| > |d'_i|$. Let $\tilde{\tilde{C}} \subset U$ be defined by $\left| d'_i \underline{u}^{\lambda_i} \right| \geq \left| \underline{u}^{\theta_i} \right|$, $1 \leq i \leq M$ and the same sign conditions as before. Then $\tilde{\tilde{C}}$ is connected and $\tilde{\tilde{C}} \subset C$.

Assume that A is of dimension 3 and has residue field $\mathbb{R}$. Let $\alpha, \beta \in$ Sper A and suppose $ht(<\alpha, \beta>) = 3$.

Let $\nu_{\alpha 0}$ be the monomial valuation defined by

$$\nu_{\alpha 0}(u_1) = \nu_\alpha(u_1) \tag{21}$$

$$\nu_{\alpha 0}(u_2) = \nu_\alpha(u_2) \tag{22}$$

$$\nu_{\alpha 0}(u_3) = \nu_\alpha(u_3). \tag{23}$$

In other words, for a polynomial $f = \sum_{\gamma \in \mathbb{N}^3} c_\gamma \underline{u}^\gamma$, we have $\nu_{\alpha 0}(f) = \min_\gamma \{\nu_\alpha(\underline{u}^\gamma) \mid c_\gamma \neq 0\}$.

Definition 5.4 *An approximate root Q for α, of complexity at most one, is said to be relevant for (α, β) if either $Q \in \{u_1, u_2, u_3\}$ or $\nu_{\alpha 0}(Q) < \nu_\alpha(<\alpha, \beta>)$.*

Note that if Q is relevant for (α, β), then Q is an approximate root for β. If, in addition $Q \notin \{u_1, u_2, u_3\}$, then $\nu_{\beta 0}(Q) < \nu_\beta(<\alpha, \beta>)$.

Let $\{Q_i\}_{4 \leq i \leq \ell}$ where $\ell \in \{3, 4, \dots, r\}$ denote the set of relevant approximate roots of complexity 1 (the case $\ell = 3$ means that the (α, β) has complexity 0).

Let $g_1, \dots, g_s \in A$ be as in the statement of the Connectedness Conjecture. Let

$$g_i = \mathbf{Q}^{\delta_i} + \sum_{j=1}^{N_i} c_{ji} \mathbf{Q}^{\delta_{ji}},\ i \in \{1, \dots, s\} \tag{24}$$

be the standard form of g_i common to α and β of level $\nu_\alpha(<\alpha, \beta>)$ (see [22], §1.3); by definition then $\mathbf{Q}^{\delta_i}$, $\mathbf{Q}^{\delta_{ji}}$ are generalized monomials in the relevant approximate roots and $\nu_\alpha(\mathbf{Q}^{\delta_i}) < \nu_\alpha(\mathbf{Q}^{\delta_{ji}})$, $j \in \{1, \dots, N_i\}$. The fact that there is only one dominant monomial $\mathbf{Q}^{\delta_i}$ is due to Theorem 4.1.

1. Let

$$C = \left\{ \delta \in \text{Sper} A \;\middle|\; \begin{array}{l} \delta \text{ is centered at } (x, y, z) \\ \nu_\delta(\mathbf{Q}^{\delta_i}) < \nu_\delta(\mathbf{Q}^{\delta_{ji}})\ \forall i \in \{1, \dots, s\},\ \forall j \in \{1, \dots, N_i\} \\ sgn_\delta(Q_q) = sgn_\alpha(Q_q) \text{ for all } Q_q \text{ appearing in } \mathbf{Q}^{\delta_i} \end{array} \right\}. \tag{25}$$

2. Let C' defined by the set of all δ, centered at (x, y, z), satisfying the inequalities

$$\left| \mathbf{Q}^{\delta_i}(\delta) \right| > N_i |\mathbf{Q}^{\delta_{ji}}(\delta)|\ \forall i \in \{1, \dots, s\},\ \forall j \in \{1, \dots, N_i\} \tag{26}$$

and the sign conditions appearing in (25).

Remark 5.5 *1. We have $\alpha, \beta \in C$.*

2. $C \cap \{g_1 \cdots g_s = 0\} = \emptyset$. Indeed, inequalities (25) imply that, for every $\delta \in C$, $g_i(\delta)$ has the same sign as $\mathbf{Q}^{\delta_i}(\delta)$; in particular, none of the g_i vanish on C.

3. To prove the Connectedness Conjecture it is sufficient to prove that α and β lie in the same connected component of C.

All the preceding remarks apply to C'.

6 The case when $ht(\mathfrak{p}) = 3$

Let $(A, \mathfrak{m})$ be an regular local ring of dimension 3 with residue field $\mathbb{R}$ contained in A. Let u_1, u_2, u_3 be a regular system of parameters of A such that $\nu_\alpha(u_1) \leq \nu_\alpha(u_2) \leq \nu_\alpha(u_3)$. Let $\alpha, \beta \in \text{Sper } A$ centered at $\mathfrak{m}$. In this case, the approximate roots of complexity 1 are binomials in u_1, u_2, u_3 (Corollary 4.2).

Lemma 6.1 *Every valuation ν admits at most three approximate roots of complexity one.*

Proof : An approximate root of complexity 1 is an irreducible binomial $\omega_1 - \omega_2$ having the property that $\nu(\omega_1 - \omega_2) > \nu(\omega_1) = \nu(\omega_2)$. We now prove that there are only three possible types of approximate roots of complexity 1 (up to exchanging the 2 monomials in order to respect the monomial ordering defined in [22], §1.2, after Definition 1.4), that means :

$$u_2^{\beta_1} u_3^{\gamma_1} - \lambda_1 u_1^{\alpha_1} \tag{27}$$

$$u_2^{\beta_2} - \lambda_2 u_1^{\alpha_2} u_3^{\gamma_2} \tag{28}$$

$$u_3^{\gamma_3} - \lambda_3 u_1^{\alpha_3} u_2^{\beta_3} \tag{29}$$

$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, with $\alpha_1, \beta_2, \gamma_3$ the smallest exponents possible.

By definition of approximate roots, the initial monomial of one approximate root cannot be divisible by the initial monomial of another. Therefore, there is at most one approximate root of each of the forms $u_2^{\beta_2} - \lambda_2 u_1^{\alpha_2} u_3^{\gamma_2}$ and $u_3^{\gamma_3} - \lambda_3 u_1^{\alpha_3} u_2^{\beta_3}$.

We claim that there is also at most one of the form $u_2^{\beta_1} u_3^{\gamma_1} - \lambda_1 u_1^{\alpha_1}$. Indeed suppose there were another one $u_2^{\beta_1'} u_3^{\gamma_1'} - \lambda_1' u_1^{\alpha_1'}$; then necessarily $\beta_1, \beta_1' < \beta_2$ and $\gamma_1, \gamma_1' < \gamma_3$. Without loss of generality we may assume that $\alpha_1 \leq \alpha_1'$. Then $\lambda_1(u_2^{\beta_1'} u_3^{\gamma_1'} - \lambda_1' u_1^{\alpha_1'}) - \lambda_1'(u_1^{\alpha_1' - \alpha_1}(u_2^{\beta_1} u_3^{\gamma_1} - \lambda_1 u_1^{\alpha_1})) = \lambda_1 u_2^{\beta_1'} u_3^{\gamma_1'} - \lambda_1' u_2^{\beta_1} u_3^{\gamma_1} u_1^{\alpha_1' - \alpha_1}$. Factoring out the greatest possible monomial, we obtain an approximate root of one of the forms (28) or (29), but with exponent of u_3 strictly less than γ_3 (respectively, exponent of u_2 strictly less than β_2), a contradiction. $\square$

Remark 6.2 *Note that, multiplying each of u_1, u_2, u_3 by a suitable non-zero element of $\mathbb{R}$, we may assume $\lambda_i = 1$ for all $i = 1, 2, 3$.*

Consider a triple of binomials Q_i, Q_j, Q_k with $Q_i = \underline{u}^{\alpha_i} - \underline{u}^{\beta_i}$, $Q_j = \underline{u}^{\alpha_j} - \underline{u}^{\beta_j}$, $Q_k = \underline{u}^{\alpha_k} - \underline{u}^{\beta_k}$, quasi-homogeneous for a certain $\mathbb{Q}$-weight ν_0, not necessarily approximate roots.

Consider the homomorphism $\sigma : \mathbb{R}[u_1, u_2, u_3] \to \mathbb{R}[[t]]$ defined by $\sigma(u_q) = t^{\nu_0(u_q)}$, $q = 1, 2, 3$. We have $Q_i, Q_j, Q_k \in \ker(\sigma)$.

Lemma 6.3 *There exists a syzygy $\omega_i Q_i + \omega_j Q_j + \omega_k Q_k = 0$ where $\omega_i, \omega_j, \omega_k$ are quasi-homogeneous polynomials in u_1, u_2, u_3 with $\omega_i, \omega_j, \omega_k \not\in \ker(\sigma)$.*

Proof : Let $\nu_0(u_i) = a_i \in \mathbb{Q}$, for $i = 1, 2, 3$. Write $\nu_0(\underline{u}^{\alpha_i}) = a_1\alpha_{i1} + a_2\alpha_{i2} + a_3\alpha_{i3}$ and the same for β_i, so that α_i, β_i belong to the plane $a_1x + a_2y + a_3z = \nu_0(\underline{u}^{\alpha_i})$.

In the same way, α_j, β_j belong to the plane $a_1x + a_2y + a_3z = \nu_0(\underline{u}^{\alpha_j})$ and α_k, β_k belong to the plane $a_1x + a_2y + a_3z = \nu_0(\underline{u}^{\alpha_k})$.

So that the vectors $v_i = \alpha_i - \beta_i,\ v_j = \alpha_j - \beta_j,\ v_k = \alpha_k - \beta_k$ belong to the plane $a_1x + a_2y + a_3z = 0$ in $\mathbb{Q}^3$. So there is a rational relation of the form $\mu_i v_i + \mu_j v_j + \mu_k v_k = 0$. Multiplying by some integer, we may choose the $\mu_i, \mu_j, \mu_k \in \mathbb{Z}$.

This gives 3 relations between the coordinates :

$$\mu_i(\alpha_{i1} - \beta_{i1}) + \mu_j(\alpha_{j1} - \beta_{j1}) + \mu_k(\alpha_{k1} - \beta_{k1}) = 0 \tag{30}$$
$$\mu_i(\alpha_{i2} - \beta_{i2}) + \mu_j(\alpha_{j2} - \beta_{j2}) + \mu_k(\alpha_{k2} - \beta_{k2}) = 0 \tag{31}$$
$$\mu_i(\alpha_{i3} - \beta_{i3}) + \mu_j(\alpha_{j3} - \beta_{j3}) + \mu_k(\alpha_{k3} - \beta_{k3}) = 0 \tag{32}$$

From which we deduce that

$$\left(\frac{(\underline{u}^{\alpha_i})^{\mu_i}}{(\underline{u}^{\beta_i})^{\mu_i}}\right) \times \left(\frac{(\underline{u}^{\alpha_j})^{\mu_j}}{(\underline{u}^{\beta_j})^{\mu_j}}\right) \times \left(\frac{(\underline{u}^{\alpha_k})^{\mu_k}}{(\underline{u}^{\beta_k})^{\mu_k}}\right) = 1 \tag{33}$$

and consequently

$$(\underline{u}^{\alpha_i})^{\mu_i}(\underline{u}^{\alpha_j})^{\mu_j}(\underline{u}^{\alpha_k})^{\mu_k} = (\underline{u}^{\beta_i})^{\mu_i}(\underline{u}^{\beta_j})^{\mu_j}(\underline{u}^{\beta_k})^{\mu_k}$$

Which we can rewrite as

$$(\underline{u}^{\alpha_i})^{\mu_i}(\underline{u}^{\alpha_j})^{\mu_j}(\underline{u}^{\alpha_k})^{\mu_k} - (\underline{u}^{\beta_i})^{\mu_i}(\underline{u}^{\beta_j})^{\mu_j}(\underline{u}^{\beta_k})^{\mu_k} = 0.$$

This last expression can be put under the following form, whatever the sign of the μ_i, μ_j, μ_k :

$$\begin{aligned}\left[(\underline{u}^{\alpha_i})^{\mu_i} - (\underline{u}^{\beta_i})^{\mu_i}\right](\underline{u}^{\alpha_j})^{\mu_j}(\underline{u}^{\alpha_k})^{\mu_k} + \left[(\underline{u}^{\alpha_j})^{\mu_j} - (\underline{u}^{\beta_j})^{\mu_j}\right](\underline{u}^{\beta_i})^{\mu_i}(\underline{u}^{\alpha_k})^{\mu_k}\\ + \left[(\underline{u}^{\alpha_k})^{\mu_k} - (\underline{u}^{\beta_k})^{\mu_k}\right](\underline{u}^{\beta_i})^{\mu_i}(\underline{u}^{\alpha_j})^{\mu_j} = 0.\end{aligned}$$

Now the relation $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1})$ applied to the first bracket shows that $\left[(\underline{u}^{\alpha_i})^{\mu_i} - (\underline{u}^{\beta_i})^{\mu_i}\right] = Q_i \times \phi_i$ where ϕ_i is a quasi-homogeneous polynomial which is clearly not in $\ker(\sigma)$. And the same with the two other brackets. This ends the proof. □

Notation : If $Q = \underline{u}^\eta - \lambda\underline{u}^\theta$ is an approximate root, we denote by Q' the expression

$$Q' = \frac{\underline{u}^\eta}{\underline{u}^\theta} - \lambda. \tag{34}$$

Let $G = \oplus_{\gamma\in\Gamma} G_\gamma$ be a graded algebra without zero divisors. The **saturation** of G, denoted by G^*, is the graded algebra

$$G^* = \{\frac{g}{h} \mid g, h \in G,\ h\ homogeneous,\ h \neq 0\}.$$

Assume that $G_\gamma \cong \mathbb{R}$ for all $\gamma \in \Gamma$. Given $\gamma \in \Gamma$ and $f, g \in G_\gamma,\ g \neq 0$, the notation $\frac{f}{g}$ will mean the unique real number λ such that $\lambda g = f$. The real number $\frac{f}{g}$ is independent of the choice of the isomorphism

$$G_\gamma \cong \mathbb{R}. \tag{35}$$

Note that the number λ can be interpreted as an element of $G_0^* \cong \mathbb{R}$.

Now let $\alpha, \beta \in \text{Sper } A$. Let $< \alpha, \beta >$ be the separating ideal. Let $\mu_\alpha = \nu_\alpha(< \alpha, \beta >)$ and $\mu_\beta = \nu_\beta(< \alpha, \beta >)$. Let Q_i, Q_j be two common approximate roots of α, β such that $Q_i, Q_j \notin < \alpha, \beta >$. Note that, since $\frac{A}{\mathfrak{m}} \cong \mathbb{R}$ and α, β are centered at $\mathfrak{m}$, the graded algebras $\text{gr}_\alpha(A)$ and $\text{gr}_\beta(A)$ satisfy the condition (35).

Definition 6.4 *We say that the two approximate roots Q_i, Q_j are (α,β)-comparable if one of the following conditions holds :*

$$\nu_\alpha(Q'_i) < \nu_\alpha(Q'_j) \text{ and } \nu_\beta(Q'_i) < \nu_\beta(Q'_j)$$

$$\nu_\alpha(Q'_i) > \nu_\alpha(Q'_j) \text{ and } \nu_\beta(Q'_i) > \nu_\beta(Q'_j)$$

$$\nu_\alpha(Q'_i) = \nu_\alpha(Q'_j),\ \nu_\beta(Q'_i) = \nu_\beta(Q'_j) \text{ and } \frac{in_\alpha Q'_i}{in_\alpha Q'_j} = \frac{in_\beta Q'_i}{in_\beta Q'_j}.$$

We say that they are strongly comparable if, up to interchanging i and j, we have

$$\nu_\alpha(Q_i) + \nu_{\alpha 0}(Q_j) < \mu_\alpha \tag{36}$$

where $\nu_{\alpha 0}$ is the monomial valuation such that $\nu_{\alpha 0}(x) = \nu_\alpha(x)$, $\nu_{\alpha 0}(y) = \nu_\alpha(y)$, $\nu_{\alpha 0}(z) = \nu_\alpha(z)$ (which implies that (36) also holds with α replaced by β).

Remark 6.5 *Note that "strongly comparable" implies "comparable". Indeed, write $Q_i = \underline{u}^{\alpha_i} - \underline{u}^{\beta_i}$, $Q_j = \underline{u}^{\alpha_j} - \underline{u}^{\beta_j}$. Without loss of generality, we may write $Q'_i = \frac{Q_i}{\underline{u}^{\alpha_i}}$, $Q'_j = \frac{Q_j}{\underline{u}^{\alpha_j}}$. By definition of strongly comparable, $Q_i\underline{u}^{\alpha_j}, Q_j\underline{u}^{\alpha_i} \not\in < \alpha, \beta >$. Thus either $\nu_\alpha(Q_i\underline{u}^{\alpha_j}) < \nu_\alpha(Q_j\underline{u}^{\alpha_i})$, $\nu_\beta(Q_i\underline{u}^{\alpha_j}) < \nu_\beta(Q_j\underline{u}^{\alpha_i})$ or $\nu_\alpha(Q_i\underline{u}^{\alpha_j}) > \nu_\alpha(Q_j\underline{u}^{\alpha_i})$, $\nu_\beta(Q_i\underline{u}^{\alpha_j}) > \nu_\beta(Q_j\underline{u}^{\alpha_i})$ or $\nu_\alpha(Q_i\underline{u}^{\alpha_j}) = \nu_\alpha(Q_j\underline{u}^{\alpha_i})$, $\nu_\beta(Q_i\underline{u}^{\alpha_j}) = \nu_\alpha(Q_j\underline{u}^{\alpha_i})$ and $\frac{in_\alpha(Q_i\underline{u}^{\alpha_j})}{in_\alpha(Q_j\underline{u}^{\alpha_i})} = \frac{in_\beta(Q_i\underline{u}^{\alpha_j})}{in_\beta(Q_j\underline{u}^{\alpha_i})}$.*

We saw that there were at most 3 approximate roots of complexity 1 for ν_α (and ν_β). Suppose there are three such approximate roots common to α and β, not in $< \alpha, \beta >$, and denote them by Q_4, Q_5, Q_6.

Lemma 6.6 *Q_4, Q_5, Q_6 are either all pairwise comparable or all pairwise incomparable.*

Proof : Assume that 2 of those roots, say Q_4 and Q_5, are comparable. Consider the syzygy $\omega_4 Q_4 + \omega_5 Q_5 + \omega_6 Q_6 = 0$ where $\omega_4, \omega_5, \omega_6$ are quasi-homogeneous polynomials, not belonging to $\ker(\sigma)$. Note that this syzygy implies that

$$\nu_{\alpha 0}(\omega_4 Q_4) = \nu_{\alpha 0}(\omega_5 Q_5) = \nu_{\alpha 0}(\omega_6 Q_6) \tag{37}$$

We will prove that Q_6 is comparable to Q_4 and so, by symmetry, the same will hold for Q_6 and Q_5. The following cases are possible :

1. $\nu_\alpha(Q'_4) < \nu_\alpha(Q'_5)$. Then $\nu_\beta(Q'_4) < \nu_\beta(Q'_5)$. Now using the syzygy and (37), we obtain that $\nu_\alpha(\omega_4 Q_4) = \nu_{\alpha 0}(\omega_4 Q_4) + \nu_\alpha(Q'_4) = \nu_{\alpha 0}(\omega_5 Q_5) + \nu_\alpha(Q'_4) < \nu_{\alpha 0}(\omega_5 Q_5) + \nu_\alpha(Q'_5) = \nu_\alpha(\omega_5 Q_5)$. So $\nu_\alpha(\omega_4 Q_4) = \nu_\alpha(\omega_6 Q_6)$ which implies that $\nu_\alpha(Q'_4) = \nu_\alpha(Q'_6)$ and, similarly, $\nu_\beta(Q'_6) = \nu_\beta(Q'_4)$.

Now, the value of $\omega_4 Q_4 + \omega_6 Q_6$ must be greater than the value of each summand because of the syzygy; in other words, the initial forms of the summands must cancel each other in the graded algebras of both ν_α and ν_β. Hence $\frac{\mathrm{in}_\alpha Q'_6}{\mathrm{in}_\alpha Q'_4} = \frac{\mathrm{in}_\beta Q'_6}{\mathrm{in}_\beta Q'_4}$, so Q_6 is comparable to Q_4.

2. $\nu_\alpha(Q'_4) > \nu_\alpha(Q'_5)$. Then $\nu_\beta(Q'_4) > \nu_\beta(Q'_5)$, so by symmetry with the previous case Q_6 is comparable to Q_5.

3. $\nu_\alpha(Q'_4) = \nu_\alpha(Q'_5)$, $\nu_\beta(Q'_4) = \nu_\beta(Q'_5)$ and

$$\frac{\mathrm{in}_\alpha Q'_4}{\mathrm{in}_\alpha Q'_5} = \frac{\mathrm{in}_\beta Q'_4}{\mathrm{in}_\beta Q'_5}. \tag{38}$$

It follows from $\nu_\alpha(Q'_4) = \nu_\alpha(Q'_5)$ that $\nu_\alpha(\omega_4 Q_4) = \nu_\alpha(\omega_5 Q_5)$ and similarly for β. Let $\gamma_\alpha = \nu_\alpha(\omega_4 Q_4)$ and $\gamma_\beta = \nu_\beta(\omega_4 Q_4)$.

Consider the natural homomorphism of graded algebras

$$\mathrm{gr}_{\alpha 0}A^* \xrightarrow{\sigma_\alpha} \mathrm{gr}_\alpha A^* \ .$$

Let $B_\alpha = \sigma_\alpha(\mathrm{gr}_{\alpha 0}A^*)$ and similarly for β.

Since there are three approximate roots of complexity 1, common to α and β, there are at least two $\mathbb{Q}$-linearly independent $\mathbb{Q}$-linear dependence relations among $\nu_\alpha(x), \nu_\alpha(y), \nu_\alpha(z)$, valid also for $\nu_\beta(x), \nu_\beta(y), \nu_\beta(z)$. Thus there is a natural graded isomorphism $gr_{\nu_{\alpha 0}}A \xrightarrow{\iota} \mathrm{gr}_{\nu_{\beta 0}}A$. Since $\iota(\ker\sigma_\alpha) = \ker(\sigma_\beta)$, the map ι induces an isomorphism $B_\alpha \to B_\beta$. We obtain the following diagram :

$$\begin{tikzcd}
\mathrm{gr}_{\alpha 0}A^* \arrow[rr, "\sigma_\alpha"] \arrow[dr] \arrow[ddd, "\iota"'] & & \mathrm{gr}_\alpha A^* \ . \\
& B_\alpha \arrow[ur, hook] \arrow[d, "\iota_B"] & \\
& B_\beta \arrow[dr, hook] & \\
\mathrm{gr}_{\beta 0}A^* \arrow[rr, "\sigma_\alpha"] \arrow[ur] & & \mathrm{gr}_\beta A^*
\end{tikzcd}$$

If $\omega_i = M_1^i + \cdots + M_{s_i}^i$ is quasi-homogeneous and not in $\ker(\sigma)$, we have, for all $j \in \{1, \ldots, s_i\}$, $\iota(\mathrm{in}_{\alpha 0}(M_j^i)) = \mathrm{in}_{\beta 0}(M_j^i)$ and $\mathrm{in}_{\alpha 0}(\omega_i) = \mathrm{in}_{\alpha 0}(M_1^i) + \cdots + \mathrm{in}_{\alpha 0}(M_{s_i}^i)$ and similarly for β. From which we deduce that $\iota(\mathrm{in}_{\alpha 0}(\omega_4)) = \mathrm{in}_{\beta 0}(\omega_4)$. On the other hand, we have $\iota(\mathrm{in}_{\alpha 0}(\underline{u}^{\alpha_4})) = \mathrm{in}_{\beta 0}(\underline{u}^{\alpha_4})$ and the same with $\underline{u}^{\alpha_4}$ replaced by $\underline{u}^{\alpha_5}$.

Now we have
$$\iota\left(\frac{\mathrm{in}_{\alpha 0}(\underline{u}^{\alpha_4}\omega_4)}{\mathrm{in}_{\alpha 0}(\underline{u}^{\alpha_5}\omega_5)}\right) = \frac{\mathrm{in}_{\beta 0}(\underline{u}^{\alpha_4}\omega_4)}{\mathrm{in}_{\beta 0}(\underline{u}^{\alpha_5}\omega_5)}.$$

Passing to the images in B_α and B_β and taking into account that $\mathrm{in}_{\alpha 0}(\underline{u}^{\alpha_4}\omega_4) \notin \ker(\sigma_\alpha)$, we obtain the equality of non-zero real numbers

$$\frac{\mathrm{in}_\alpha(\underline{u}^{\alpha_4}\omega_4)}{\mathrm{in}_\alpha(\underline{u}^{\alpha_5}\omega_5)} = \frac{\mathrm{in}_\beta(\underline{u}^{\alpha_4}\omega_4)}{\mathrm{in}_\beta(\underline{u}^{\alpha_5}\omega_5)}. \tag{39}$$

Multiplying the equations (38) and (39), we obtain

$$\frac{\mathrm{in}_\alpha(\omega_4)\mathrm{in}_\alpha(\underline{u}^{\alpha_4})\mathrm{in}_\alpha(Q_4')}{\mathrm{in}_\alpha(\omega_5)\mathrm{in}_\alpha(\underline{u}^{\alpha_5})\mathrm{in}_\alpha(Q_5')} = \frac{\mathrm{in}_\beta(\omega_4)\mathrm{in}_\beta(\underline{u}^{\alpha_4})\mathrm{in}_\beta(Q_4')}{\mathrm{in}_\beta(\omega_5)\mathrm{in}_\beta(\underline{u}^{\alpha_5})\mathrm{in}_\beta(Q_5')}. \tag{40}$$

In other words,
$$\frac{\mathrm{in}_\alpha(\omega_4 Q_4)}{\mathrm{in}_\alpha(\omega_5 Q_5)} = \frac{\mathrm{in}_\beta(\omega_4 Q_4)}{\mathrm{in}_\beta(\omega_5 Q_5)}.$$

3a. $\mathrm{in}_\alpha\omega_4 Q_4 + \mathrm{in}_\alpha\omega_5 Q_5 = 0$. Then $\mathrm{in}_\beta\omega_4 Q_4 + \mathrm{in}_\beta\omega_5 Q_5 = 0$. Then $\nu_\alpha(Q_6') > \nu_\alpha(Q_4')$ and $\nu_\beta(Q_6') > \nu_\beta(Q_4')$, so Q_6 is comparable to Q_4.

3b. $\mathrm{in}_\alpha\omega_4 Q_4 + \mathrm{in}_\alpha\omega_5 Q_5 \neq 0$. Then $\mathrm{in}_\beta\omega_4 Q_4 + \mathrm{in}_\beta\omega_5 Q_5 \neq 0$. Then, using (37), $\nu_\alpha(Q_6') = \nu_\alpha(Q_4')$ and $\nu_\beta(Q_6') = \nu_\beta(Q_4')$. Write the syzygy in the form $\omega_4' Q_4' + \omega_5' Q_5' + \omega_6' Q_6' = 0$ where $\nu_\alpha(\omega_4') = \nu_\alpha(\omega_5') = \nu_\alpha(\omega_6')$ where, for $i = 4, 5, 6$, $\omega_i' = \omega_i \underline{u}^{\alpha_i}$. Now,

$$\frac{\mathrm{in}_\alpha Q_6'}{\mathrm{in}_\alpha Q_4'} = \frac{-\mathrm{in}_\alpha \frac{\omega_4'}{\omega_6'} Q_4' - \mathrm{in}_\alpha \frac{\omega_5'}{\omega_6'} Q_5'}{\mathrm{in}_\alpha Q_4'} = -\mathrm{in}_\alpha \frac{\omega_4'}{\omega_6'} - \mathrm{in}_\alpha \frac{\omega_5'}{\omega_6'} \frac{\mathrm{in}_\alpha Q_5'}{\mathrm{in}_\alpha Q_4'} = -\mathrm{in}_\beta \frac{\omega_4'}{\omega_6'} - \mathrm{in}_\beta \frac{\omega_5'}{\omega_6'} \frac{\mathrm{in}_\beta Q_5'}{\mathrm{in}_\beta Q_4'} = \frac{\mathrm{in}_\beta Q_6'}{\mathrm{in}_\beta Q_4'}$$

and so again Q_6 is comparable to Q_4.

From now on, assume that A is excellent.

6.1 Some of u_1, u_2, u_3 belong to the separating ideal

Assume that $\{u_1, u_2, u_3\} \cap <\alpha, \beta> \neq \emptyset$. The case $u_1, u_2, u_3 \in <\alpha, \beta>$ is trivial since then $<\alpha, \beta> = \mathfrak{m}$.

If $u_1 \notin <\alpha, \beta>$ and $u_2, u_3 \in <\alpha, \beta>$, then the only relevant approximate roots appearing in (24) and (26) are u_1, u_2, u_3. Then all the inequalities defining C' are monomial in u_1, u_2, u_3. So C' is connected by Corollary 5.2.

Finally, suppose that $u_1, u_2 \notin <\alpha, \beta>$, $u_3 \in <\alpha, \beta>$. Then all the approximate roots belong to $\{u_3\} \cup \mathbb{R}[u_1, u_2]$.

After a suitable sequence of affine monomial blowings up $A \to A'$ such that Sper A' contains the common center $\mathfrak{m}'$ of α' and β', $\mathfrak{m}'$ has a regular system of parameters u_1', u_2', u_3' such that all the approximate roots are monomials in u_1', u_2', u_3' up to multiplication by units of A'. Let ξ' be the unique point of Sper A' with support $\mathfrak{m}'$.

Let π : Sper $A' \to$ Sper A the induced map of real spectra. Write $\mathbf{Q}^{\delta_i} = \underline{u}'^{\delta_i'} v_i$, $\mathbf{Q}^{\delta_{ji}} = \underline{u}'^{\delta_{ji}'} v_{ji}$ with $v_i, v_{ji} \in A' \setminus \mathfrak{m}'$. If C is as in (25), the set $\pi^{-1}(C)$ contains the set

$$\tilde{C} = \left\{ \begin{array}{l} \delta \text{ is centered at } \mathfrak{m}' \\ \nu_\delta(\underline{u}'^{\delta_i'}) < \nu_\delta(\underline{u}'^{\delta_{ji}'}),\ i \in \{1, \dots, s\}, j \in \{1, \dots, N_i\} \\ sgn_\delta u_\ell' = sgn_\alpha u_\ell' \text{ for all } u_\ell' \text{ appearing in } \underline{u}'^{\delta_i'} \text{ for some } i \end{array} \right\}.$$

Take $d \in \mathbb{R}$ such that $d > \max\limits_{1 \leq i \leq s} N_i \times \max\limits_{\substack{1 \leq i \leq s \\ 1 \leq j \leq N_i}} \dfrac{|v_{ji}(\xi')|}{|v_i(\xi')|}$.

Let

$$\tilde{\tilde{C}} = \left\{ \begin{array}{l} \delta \text{ is centered at } \mathfrak{m}' \\ |\underline{u}'^{\delta_i'}(\delta)| > d|\underline{u}'^{\delta_{ji}'}(\delta)|\ i \in \{1, \dots, s\}, j \in \{1, \dots, N_i\} \\ sgn_\delta u_\ell' = sgn_\alpha u_\ell' \text{ for all } u_\ell' \text{ appearing in } \underline{u}'^{\delta_i'} \text{ for some } i \end{array} \right\}.$$

$\tilde{\tilde{C}}$ is connected by Corollary 5.2.

Then $\alpha', \beta' \in \tilde{\tilde{C}}$, hence $\alpha, \beta \in \pi(\tilde{\tilde{C}})$, $\pi(\tilde{\tilde{C}})$ is connected and contained in $\{g_1 \cdots g_s \neq 0\}$. This completes the proof in the case when $\{u_1, u_2, u_3\} \cap <\alpha, \beta> \neq \emptyset$.

From now on, we assume $\{u_1, u_2, u_3\} \cap <\alpha, \beta> = \emptyset$ and that, unless otherwise specified, there are 3 relevant approximate roots Q_4, Q_5, Q_6.

6.2 All the approximate roots are pairwise comparable

Without loss of generality, assume that $\nu_\alpha(Q_6') \leq \nu_\alpha(Q_4')$, $\nu_\alpha(Q_6') \leq \nu_\alpha(Q_5')$ and $u_1(\alpha) > 0, u_2(\alpha) > 0, u_3(\alpha) > 0, Q_6(\alpha) > 0$.

Write

$$Q_6 = -\frac{\omega_4}{\omega_6} Q_4 - \frac{\omega_5}{\omega_6} Q_5. \tag{41}$$

Assume that $\frac{\omega_5}{\omega_6} Q_5 >_{\alpha,\beta} 0$ (if the opposite holds, a similar reasoning applies). Then $Q_6 >_{\alpha,\beta} 0 \Rightarrow |\frac{\omega_4}{\omega_6} Q_4| >_{\alpha,\beta} |\frac{\omega_5}{\omega_6} Q_5|$. There exists $\epsilon > 0$ in $\mathbb{R}$ such that

$$(1 - \epsilon)|\frac{\omega_4}{\omega_6} Q_4| >_{\alpha,\beta} |\frac{\omega_5}{\omega_6} Q_5|. \tag{42}$$

Then

$$|Q_6| >_{\alpha,\beta} \epsilon |\frac{\omega_4}{\omega_6} Q_4|. \tag{43}$$

We now describe the connected set required in the Connectedness Conjecture which we will define by inequalities among certain generalized monomials and sign conditions on the Q_i. Let $g_1, \dots, g_s \in A$ be as in the statement of the conjecture. Let

$$g_i = \mathbf{Q}^{\delta_i} + \sum_{j=1}^{N_i} c_{ji} \mathbf{Q}^{\delta_{ji}},\ i \in \{1, \dots, s\} \tag{44}$$

be the standard form of g_i.

For each $i \in \{1, \ldots, s\}$, in the sum $\sum_{j=1}^{N_i} c_{ji}\mathbf{Q}^{\delta_{ji}}$ replace Q_6 by the right hand side of (41) and write the result as a sum of generalized monomials (with possibly negative exponents) in u_1, u_2, u_3, Q_4, Q_5 :

$$\sum_{j=1}^{N'_i} c'_{ji}\mathbf{Q}^{\delta'_{ji}}.$$

In each generalized monomial $\mathbf{Q}^{\delta_i}$, replace Q_6 by $\epsilon\frac{\omega_4}{\omega_6}Q_4$ and let $c'_i\mathbf{Q}^{\delta'_i}$ be the resulting generalized monomial.

Let D be the subset of Sper A consisting of points δ such that

$$|c'_i\mathbf{Q}^{\delta'_i}(\delta)| > \frac{1}{N'_i}|c'_{ji}\mathbf{Q}^{\delta'_{ji}}(\delta)| \quad i \in \{1, \ldots, s\},\ j \in \{1, \ldots, N_i\} \tag{45}$$

$$(1-\epsilon)|\frac{\omega_4}{\omega_6}Q_4(\delta)| > |\frac{\omega_5}{\omega_6}Q_5(\delta)| \tag{46}$$

$$sgn(u_1(\delta)) = sgn(u_1(\alpha)) \tag{47}$$

$$sgn(u_2(\delta)) = sgn(u_2(\alpha)) \tag{48}$$

$$sgn(u_3(\delta)) = sgn(u_3(\alpha)) \tag{49}$$

$$sgn(Q_4(\delta)) = sgn(Q_4(\alpha)) \tag{50}$$

$$sgn(Q_5(\delta)) = sgn(Q_5(\alpha)) \tag{51}$$

By definition of standard form, $\nu_\alpha(\mathbf{Q}^{\delta_i}) < \nu_\alpha(\mathbf{Q}^{\delta_{ji}})$ for all i, j and similarly for ν_β. By (43), this implies that $\nu_\alpha(\mathbf{Q}^{\delta'_i}) < \nu_\alpha(\mathbf{Q}^{\delta'_{ji}})$ for all i, j and the same for ν_β. Thus inequalities (45) hold for $\delta = \alpha$ and $\delta = \beta$. This proves that $\alpha,\ \beta \in D$. The polynomials g_i have constant sign on D for all i because the inequalities ensure that the sign of g_i is determined by the sign of its dominant monomial $\mathbf{Q}^{\delta_i}$. With these conditions, using (41), we see that the signs of both Q_6 and $\mathbf{Q}^{\delta_i}$ are constant on D.

It remains to prove that α and β belong to the same connected component of D Let $A \to A'$ be a finite sequence of affine monomial blowings up such that Sper A' contains the common center m' of α' and β' and there is a regular system of parameters (x', y', z') at m' such that x' is a monomial in u_1, u_2, u_3, $y' = Q'_4$ and $z' = Q'_5$.

For each inequality

$$|c\mathbf{Q}^\epsilon(\delta)| < |d\mathbf{Q}^\gamma(\delta)| \tag{52}$$

appearing in the definition of D, there exist $\epsilon'_x, \epsilon'_y, \epsilon'_z, \gamma'_x, \gamma'_y, \gamma'_z \in \mathbb{Z}$ and elements $u, v \in A'_{m'} \setminus m'A'_{m'}$ such that $c\mathbf{Q}^\epsilon = ux'^{\epsilon'_x}y'^{\epsilon'_y}z'^{\epsilon'_z}$ and $d\mathbf{Q}^\gamma = vx'^{\gamma'_x}y'^{\gamma'_y}z'^{\gamma'_z}$. Take positive constants $\tilde{u}, \tilde{v} \in R$ such that $|u| < \tilde{u}$ and $|v| > \tilde{v}$. Then, for any $\delta' \in$ Sper $A'_{m'}$, the inequality

$$|\tilde{u}x'^{\epsilon'_x}(\delta')y'^{\epsilon'_y}(\delta')z'^{\epsilon'_z}(\delta')| < |\tilde{v}x'^{\gamma'_x}(\delta')y'^{\gamma'_y}(\delta')z'^{\gamma'_z}(\delta')| \tag{53}$$

implies the inequality (52) where δ is the image of δ' in Sper A.

Since

$$\nu_\alpha(x'^{\epsilon'_x}y'^{\epsilon'_y}z'^{\epsilon'_z}) > \nu_\alpha(x'^{\gamma'_x}y'^{\gamma'_y}z'^{\gamma'_z}) \tag{54}$$

and similarly for β, the inequalities (53) hold for both $\delta' = \alpha'$ and $\delta' = \beta'$.

Let D' be the subset of Sper $A'_{m'}$ defined by all the resulting inequalities of the form (53) and the sign conditions

$$sgn(x'(\delta')) = sgn(x'(\alpha')) \tag{55}$$

$$sgn(y'(\delta')) = sgn(y'(\alpha')) \tag{56}$$

$$sgn(z'(\delta')) = sgn(z'(\alpha')). \tag{57}$$

The set D' is connected by Corollary 5.2. Its image in Sper A is connected, contains α and β and is contained in the set $\{g_1 \cdots g_s \neq 0\}$. This completes the proof of the Connectedness Conjecture in the case when Q_4, Q_5, Q_6 are pairwise comparable.

Remark 6.7 *The same method works to prove the Connectedness conjecture also in the case when $\nu_\beta(Q'_4) = \nu_\beta(Q'_5) = \nu_\beta(Q'_6)$ and $\nu_\alpha(Q'_4) = \nu_\alpha(Q'_5) < \nu_\alpha(Q'_6)$.*

6.3 The case when there are only one or two relevant approximate roots

Assume that there are exactly 2 approximate roots $Q_4, Q_5 \notin <\alpha, \beta>$, common to α and β. We proceed as in the case of 3 comparable approximate roots. This means that, after a suitable sequence of affine monomials blowings up $A \to A'$ such that Sper A' contains the common center m' of α' and β', m' has a regular system of parameters x', y', z' such that x' is a monomial in u_1, u_2, u_3, $y' = Q'_4$, $z' = Q'_5$ (remember the notation (34)).

Then as above, we replace the inequalities of type (52) by inequalities involving only monomials as in (53). Once again we apply the Corollary 5.2 to ensure the existence of a set $\mathcal{C}$ as required.

The case with only one relevant approximate root, Q_4, is more difficult.

Claim. There exists a connected subset $\tilde{C}$ of C, which we will describe explicitly, containing α and β.

Proof : First, we consider the special case when A is the localization of the polynomial ring $A = \mathbb{R}[x, y, z]_{(x,y,z)}$. Let $A \to A_1 \to \cdots \to A_i$ be a finite sequence of affine monomial blowings up with respect to α (see [21], Proposition 6.1) such that $\underline{u}^{\alpha_4 - \beta_4} \in A_i$ where $\underline{u} = (x, y, z)$. Let $z_i = \underline{u}^{\alpha_4 - \beta_4} - 1$. By construction, A_i is the localization of a polynomial ring of the form $\mathbb{R}[x_i, y_i, z_i]$, where x_i, y_i are monomials in x, y, z and $z_i = \underline{u}^{\alpha_4 - \beta_4} - 1$, by the multiplicative set $\mathbb{R}[x, y, z] \setminus (x, y, z)$. Let $\underline{u}_i = (x_i, y_i, z_i)$.

In A_i the inequalities (26) can be rewritten as

$$|(c_i + z_i f_i)\underline{u}_i^{\gamma_i}| < \left|(d_i + z_i h_i)\underline{u}_i^{\delta_i}\right| \text{ where } c_i, d_i \in \mathbb{R},\ c_i d_i \neq 0,\ f_i, h_i \in \mathbb{R}[x_i, y_i, z_i]. \tag{58}$$

Let $\tilde{C} \subset$ Sper A_i be the set defined by the stronger inequalities

$$|2c_i \underline{u}_i^{\gamma_i}| \leq \left|\frac{1}{2} d_i \underline{u}_i^{\delta_i}\right| \tag{59}$$

and the sign conditions $sgn(x_i(\delta)) = sgn(x_i(\alpha))$, $sgn(y_i(\delta)) = sgn(y_i(\alpha))$, $sgn(z_i(\delta)) = sgn(z_i(\alpha))$. The set $\tilde{C}$ contains the strict transforms of α and β.

Let D be the subset of Sper $\mathbb{R}[x_i, y_i, z_i]$ defined by the inequalities (59) and the same sign conditions as above.

Using the cartesian diagram

$$\begin{tikzcd}
\text{Sper } \mathbb{R}[x, y, z] & \text{Sper } \mathbb{R}[x_i, y_i, z_i] \arrow[l] \\
\text{Sper } \mathbb{R}[x, y, z]_{(x,y,z)} \arrow[u, hook] \arrow[ur, phantom, "\square"] & \text{Sper } \mathbb{R}[x_i, y_i, z_i]_S \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

where $S = \mathbb{R}[x, y, z] \setminus (x, y, z)$, we can identify $\tilde{C}$ with

$$\bigcap_{N=1}^{\infty} D \cap \{\delta \in \text{Sper } \mathbb{R}[x_i, y_i, z_i]\ ;\ |x(\delta)| \leq \frac{1}{N},\ |y(\delta)| \leq \frac{1}{N},\ |z(\delta)| \leq \frac{1}{N}\}.$$

By Lemma 4.1 of [21], we have that $D \cap \{\delta \in \text{Sper } \mathbb{R}[x_i, y_i, z_i] \; ; \; |x(\delta)| \leq \frac{1}{N}, \; |y(\delta)| \leq \frac{1}{N}, \; |z(\delta)| \leq \frac{1}{N}\}$ is connected, so applying Lemma 7.1 of [21], we deduce that the intersection is a non empty closed connected set, so $\tilde{C}$ is connected and contains the strict transforms of α and β, hence its image in Sper A is the desired connected set. This completes the proof when $A = \mathbb{R}[x, y, z]_{(x,y,z)}$. The general case now follows from Theorem 5.1.

6.4 The approximate roots are pairwise incomparable

Lemma 6.8 *At least two of Q_4, Q_5, Q_6 have ν_α-value strictly greater than $\frac{\mu_\alpha}{2}$ (and similarly for ν_β-value).*

Proof : This is an immediate consequence of Definition 6.4 and Remark 6.5. $\square$

Without loss of generality, assume that

$$\nu_\alpha(Q_4) \leq \nu_\alpha(Q_5) \leq \nu_\alpha(Q_6). \tag{60}$$

Then

$$\nu_\alpha(Q_5) > \frac{\mu_\alpha}{2}, \; \nu_\alpha(Q_6) > \frac{\mu_\alpha}{2}. \tag{61}$$

Proposition 6.9 *Consider a generalized monomial $\mathbf{Q}^\gamma$ divisible by one of Q_4Q_5, Q_4Q_6, Q_5Q_6, Q_5^2, Q_6^2.*
Then (a) $\mathbf{Q}^\gamma \in < \alpha, \beta >$
(b) $\mathbf{Q}^\gamma$ belongs to the ideal generated by all the generalized monomials belonging to $< \alpha, \beta >$ and not divisible by any of $Q_5^2, Q_6^2, Q_4Q_5, Q_4Q_6, Q_5Q_6$.

Proof : (a) If $Q_5^2 \mid \mathbf{Q}^\gamma$ or $Q_6^2 \mid \mathbf{Q}^\gamma$, the result follows immediately from (61).
If $Q_4Q_5 \mid \mathbf{Q}^\gamma$, $Q_4Q_6 \mid \mathbf{Q}^\gamma$ or $Q_5Q_6 \mid \mathbf{Q}^\gamma$, the result follows from Definition 6.4 and Lemma 6.5.

(b) By pairwise incomparability and (60), we have $\nu_\alpha(Q_5) + \nu_{0\alpha}(Q_5) \geq \mu_\alpha$ and similarly for β. As well,

$$\nu_\alpha(Q_6) + \nu_{0\alpha}(Q_6) \geq \mu_\alpha. \tag{62}$$

Suppose, for example, $Q_5^2 \mid \mathbf{Q}^\gamma$. Write $Q_5 = \omega - \epsilon$ where ω, ϵ are monomials in x, y, z. Then $\mathbf{Q}^\gamma$ belongs to the ideal generated by $Q_5\epsilon$, $Q_5\omega$ and by (62), $Q_5\epsilon, Q_5\omega \in < \alpha, \beta >$. The cases when $\mathbf{Q}^\gamma$ is divisible by Q_6^2, Q_4Q_5, Q_4Q_6, Q_5Q_6 are handled similarly. $\square$

We now describe the connected set required in the Connectedness Conjecture by inequalities on the size of certain generalized monomials and sign conditions on the Q_i. Let $g_1, \ldots, g_s \in A$ be as in the statement of the conjecture. Let

$$g_i = \mathbf{Q}^{\delta_i} + \sum_{j=1}^{N_i} c_{ji} \mathbf{Q}^{\delta_{ji}}, \; i \in \{1, \ldots, s\} \tag{63}$$

be the standard form of g_i of level $\nu_\alpha(< \alpha, \beta >$.

Let $\mathcal{S}$ be a finite set of generalized monomials, not divisible by Q_5^2, Q_6^2, Q_4Q_5, Q_4Q_6, Q_5Q_6, belonging to $< \alpha, \beta >$, which generate $< \alpha, \beta >$. In addition, we require all the monomials $\mathbf{Q}^\lambda \in \mathcal{S}$ to have the following property : if $Q_4 \mid \mathbf{Q}^\lambda$ then

$$\nu_\alpha(\mathbf{Q}^\lambda) - \nu_\alpha(Q_4) + \nu_{\alpha 0}(Q_4) < \mu_\alpha \tag{64}$$

and similarly for Q_5 and Q_6.

Let $\mathcal{T}$ be the set of all the generalized monomials not belonging to $< \alpha, \beta >$.

Let C be the subset of Sper A consisting of points δ such that

$$\nu_\delta(\mathbf{Q}^\gamma) < \nu_\delta(\mathbf{Q}^\lambda) \tag{65}$$
$$\nu_\delta(\mathbf{Q}^\theta) = \nu_\delta(\mathbf{Q}^\eta) \tag{66}$$
$$sgn(u_1(\delta)) = sgn(u_1(\alpha)) \tag{67}$$
$$sgn(u_2(\delta)) = sgn(u_2(\alpha)) \tag{68}$$
$$sgn(u_3(\delta)) = sgn(u_3(\alpha)) \tag{69}$$
$$sgn(Q_4(\delta)) = sgn(Q_4(\alpha)) \tag{70}$$
$$sgn(Q_5(\delta)) = sgn(Q_5(\alpha)) \tag{71}$$
$$\mathrm{sgn}(Q_6(\delta)) = sgn(Q_6(\alpha)) \tag{72}$$

where (i) $\mathbf{Q}^\theta, \mathbf{Q}^\eta$ run over all the pairs of elements of $\mathcal{T}$ satisfying (66) for $\delta = \alpha$ and $\delta = \beta$,

(ii) $\mathbf{Q}^\gamma, \mathbf{Q}^\lambda$ run over all the pairs of generalized monomials such that $\mathbf{Q}^\gamma \in \mathcal{T}$, $\mathbf{Q}^\lambda \in \mathcal{T} \cup \mathcal{S}$ and (65) holds for $\delta = \alpha$ and $\delta = \beta$.

Note that the definition of C implies that for all $\delta \in C$, the binomials Q_4, Q_5, Q_6 are approximate roots for the valuation ν_δ.

All points $\delta \in C$ share, by definition of C, the same approximate roots Q_4, Q_5, Q_6. This implies that the $\dim_{\mathbb{Q}}(\sum_{j=1}^{3} \mathbb{Q}\nu_\delta(u_i)) = 1$, for all $\delta \in C$ (in particular, for $\delta = \alpha$ and $\delta = \beta$). Moreover, there exist $r, s \in \mathbb{Q}$ such that $\nu_\delta(u_2) = r\nu_\delta(u_1)$ and $\nu_\delta(u_3) = s\nu_\delta(u_1)$ for all $\delta \in C$. Then each equality or inequality of (65), (66) may be written in a form containing only $\nu(u_1)$ and $\nu(Q_4), \nu(Q_5), \nu(Q_6)$. As $\nu_{\delta 0}(\mathbf{Q}^\lambda)$ can be written purely in terms of $\nu(u_1)$ and as $\nu(Q_\ell) = \nu_0(Q_\ell) + \nu(Q'_\ell)$, any relation of the form (65) or (66) may be written in terms of $\nu(u_1), \nu(Q'_4), \nu(Q'_5), \nu(Q'_6)$.

Proposition 6.10 *C contains a point ϵ such that $\nu_\epsilon(Q'_4) = \nu_\epsilon(Q'_5) = \nu_\epsilon(Q'_6)$.*

Without loss of generality, assume that

$$\nu_\alpha(Q'_4) = \nu_\alpha(Q'_5) < \nu_\alpha(Q'_6) \quad \text{and} \tag{73}$$
$$\nu_\beta(Q'_4) = \nu_\beta(Q'_6) < \nu_\alpha(Q'_5) \tag{74}$$

Replacing α by α' lying in C such that

$$\nu_{\alpha'}(Q'_4) = \nu_{\alpha'}(Q'_5) < \nu_{\alpha'}(Q'_6) \tag{75}$$

and $\Gamma_{\alpha'} \subset \mathbb{R}$ does not change the problem and similarly for β. From now on, we will assume that $\Gamma_\alpha \subset \mathbb{R}$ and $\Gamma_\beta \subset \mathbb{R}$.

Let

$$\phi : \left\{ \delta \in \text{Sper } A \;\middle|\; \begin{array}{l} \Gamma_\delta \subset \mathbb{R} \\ \delta \text{ centered in } \mathfrak{m} \\ Q_4, Q_5, Q_6 \text{ are approximate} \\ \text{roots for } \delta. \end{array} \right\} \rightarrow \begin{array}{c} \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \\ \mid a_1, a_2, a_3, a_4 > 0\} \end{array} \tag{76}$$

be the map defined by $\phi(\delta) = (\nu_\delta(u_1), \nu_\delta(Q'_4), \nu_\delta(Q'_5), \nu_\delta(Q'_6))$.

Lemma 6.11 *A point $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ is in the image of ϕ if and only if one of the following conditions holds*

$$a_2 = a_3 \le a_4, \;\; a_2 = a_4 \le a_3, \;\; a_3 = a_4 \le a_2. \tag{77}$$

Proof : Write the syzygy in the form $\omega'_4 Q'_4 + \omega'_5 Q'_5 + \omega'_6 Q'_6 = 0$. The "only if" part follows from this.

"If" : Suppose, for example, that $a_2 = a_3 \le a_4$. Consider a sequence of blowings up $A \to A'$ such that A' has a maximal ideal m' with a regular system of parameters (x', y', z')

such that $Q'_4 = y', Q'_6 = z'$ and u_1, u_2, u_3 are monomials in x', up to multiplication by units of A'. Write $u_1 = x'^{\gamma} v$ with $\gamma \in \mathbb{N}_+^*$, $v \in A' \setminus m'$. Take a point δSper A such that $\nu_\delta(x') = \frac{a_1}{\gamma}$, $\nu_\delta(y') = a_2$, $\nu_\delta(z') = a_4$. Then δ is centered in $\mathfrak{m}$ and has Q_4, Q_5, Q_6 as approximate roots. this proves that $(a_1, a_2, a_3, a_4) \in Im(\phi)$.

Next, we reduce the problem to the case when each of the inequalities (65) and equalities (66) involves at most one of Q_4, Q_5, Q_6 (possibly raised to some power). Namely let $h_1, \dots, h_p$ be the complete list of inequalities (65) and equalities (66). We order the h_i in such a way that all the inequalities-equalities involving at most one of Q_4, Q_5, Q_6 come first and those involving at least 2 of Q_4, Q_5, Q_6 come later. In each of the above two lists, we order the inequalities-equalities by the value of the left-hand side.

Lemma 6.12 *Assume Proposition 6.10 is true in the special case when each h_i contains at most one of Q_4, Q_5, Q_6. Then it is true in general.*

Proof : We argue by contradiction. Suppose that Proposition 6.10 is false for $h_1, \dots, h_p$. Without loss of generality, we may assume that Proposition is true for $h_1, \dots, h_{p-1}$. Let $\tilde{C} \supset C$ be the set defined by the same conditions as C except for h_p. By assumptions, $\tilde{C}$ contains a point ϵ such that $\nu_\epsilon(Q'_4) = \nu_\epsilon(Q'_5) = \nu_\epsilon(Q'_6)$.

In the following formulae, the notation

$$\{\nu_\delta(\mathbf{Q}^\gamma) = \nu_\delta(\mathbf{Q}^\lambda)\} \tag{78}$$

means "the set of all the points of $\mathbb{R}^4$ of the form $(\nu_\delta(u_1), \nu_\delta(Q'_4), \nu_\delta(Q'_5), \nu_\delta(Q'_6))$ where δ satisfies (78)". This notation makes sense because because $\nu_\delta(\mathbf{Q}^\gamma)$ and $\nu_\delta(\mathbf{Q}\lambda)$ are completely determined by $(\nu_\delta(u_1), \nu_\delta(Q'_4), \nu_\delta(Q'_5), \nu_\delta(Q'_6))$. The set (78) is contained in a hyperplane H of $\mathbb{R}^4$, defined by a linear equation with rational coefficients, and contains the subset of H satisfying the conditions (77).

If h_p is a strict inequality, write h_p in the form $\nu_\delta(\mathbf{Q}^\gamma) < \nu_\delta(\mathbf{Q}^\lambda)$ and consider two segments in $\mathbb{R}^4$

$$[(\nu_\alpha(u_1), \nu_\alpha(Q'_4), \nu_\alpha(Q'_5), \nu_\alpha(Q'_6)), (\nu_\epsilon(u_1), \nu_\epsilon(Q'_4), \nu_\epsilon(Q'_5), \nu_\epsilon(Q'_6))] \tag{79}$$

$$[(\nu_\beta(u_1), \nu_\beta(Q'_4), \nu_\beta(Q'_5), \nu_\beta(Q'_6)), (\nu_\epsilon(u_1), \nu_\epsilon(Q'_4), \nu_\epsilon(Q'_5), \nu_\epsilon(Q'_6))]. \tag{80}$$

Since $\nu_\epsilon(Q'_4) = \nu_\epsilon(Q'_5) = \nu_\epsilon(Q'_6)$ and the left endpoint of

$$[(\nu_\alpha(u_1), \nu_\alpha(Q'_4), \nu_\alpha(Q'_5), \nu_\alpha(Q'_6)), (\nu_\epsilon(u_1), \nu_\epsilon(Q'_4), \nu_\epsilon(Q'_5), \nu_\epsilon(Q'_6))]$$

satisfies the conditions (77), so does every point of that interval. The same holds for the interval

$$[(\nu_\beta(u_1), \nu_\beta(Q'_4), \nu_\beta(Q'_5), \nu_\beta(Q'_6)), (\nu_\epsilon(u_1), \nu_\epsilon(Q'_4), \nu_\epsilon(Q'_5), \nu_\epsilon(Q'_6))].$$

Since $\epsilon \notin C$, the following intersections are non empty; each of them consists of one point

$$[(\nu_\alpha(u_1), \nu_\alpha(Q'_4), \nu_\alpha(Q'_5), \nu_\alpha(Q'_6)), (\nu_\epsilon(u_1), \nu_\epsilon(Q'_4), \nu_\epsilon(Q'_5), \nu_\epsilon(Q'_6))] \tag{81}$$
$$\cap \left\{\nu_\delta(\mathbf{Q}^\gamma) = \nu_\delta(\mathbf{Q}^\lambda)\right\} =: \{(a_1, a_2, a_3, a_4)\} \tag{82}$$

and

$$[(\nu_\beta(u_1), \nu_\beta(Q'_4), \nu_\beta(Q'_5), \nu_\beta(Q'_6)), (\nu_\epsilon(u_1), \nu_\epsilon(Q'_4), \nu_\epsilon(Q'_5), \nu_\epsilon(Q'_6))] \tag{83}$$
$$\cap \{\nu_\delta(\mathbf{Q}^\gamma) = \nu_\delta(\mathbf{Q}^\lambda)\} =: \{(b_1, b_2, b_3, b_4)\}. \tag{84}$$

Take points $\alpha_0 \in \phi^{-1}((a_1, a_2, a_3, a_4))$ and $\beta_0 \in \phi^{-1}((b_1, b_2, b_3, b_4))$.

In particular

$$\nu_{\alpha_0}(Q'_4) = \nu_{\alpha_0}(Q'_5) \leq \nu_{\alpha_0}(Q'_6) \quad \text{and} \tag{85}$$
$$\nu_{\beta_0}(Q'_4) = \nu_{\beta_0}(Q'_6) \leq \nu_{\beta_0}(Q'_5) \tag{86}$$

and at least one of the two inequalities is strict. Note that $\alpha_0 \neq \beta_0$. Suppose the inequality (85) is strict. Let h_{p_0} be the equality $\nu_\delta(\mathbf{Q}^\gamma) = \nu_\delta(\mathbf{Q}^\lambda)$.

If h_p is an equality, put $\alpha_0 = \alpha$, $\beta_0 = \beta$ and let $h_{p_0} = h_p$.

We can now contradict (85), (86) as follows. Suppose, for example, that h_{p_0} has the form

$$\nu_\delta(Q_4^a\omega) = \nu_\delta(Q_5\eta) \tag{87}$$

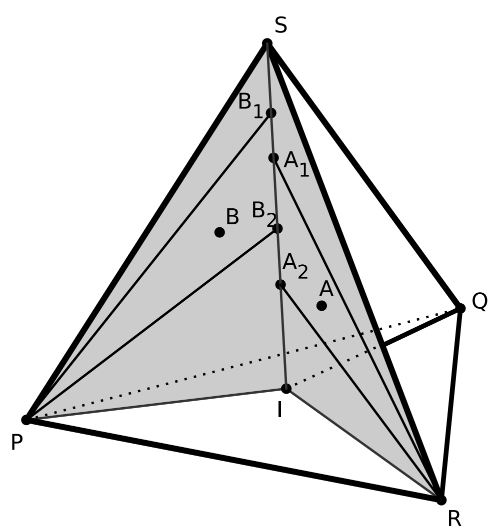
$a \geq 1$ and write $Q_4 = \epsilon_4 - \omega_4$, $Q_5 = \epsilon_5 - \omega_5$.

Then $Q_4^{a-1}\omega\epsilon_4$ and $\epsilon_5\eta$ do not belong to $< \alpha, \beta >$ and the relation between $\nu_\alpha(Q_4^{a-1}\omega\epsilon_4)$ and $\nu_\alpha(\epsilon_5\eta)$ (which may be $<, =$ or $>$) belongs to the list $h_1, \dots, h_{p-1}$. Therefore, this relation is the same for α_0 and β_0. From this, it follows that the relation between $\nu_\delta(Q_4) - \nu_{\delta 0}(Q_4) = \nu_\delta(Q_4')$ and $\nu_\delta(Q_5) - \nu_{\delta 0}(Q_5) = \nu_\delta(Q_5')$ (which may be $<, =$ or $>$) is the same for $\delta = \alpha_0$ and for $\delta = \beta_0$. This contradicts (85), (86). $\square$

We now need the following geometric lemma.

Lemma 6.13 - ***Tetrahedron Lemma*** - *Consider 4 points of $\mathbb{R}^3$, P, Q, R, S, not lying in the same plane. Then they define an affine basis of $\mathbb{R}^3$ and let (u, v, w, t) be the barycentric coordinates with respect to this basis (so $u+v+w+t = 1$). Let A be a point with coordinates (u_1, v_1, w_1, t_1) such that $u_1, v_1, w_1, t_1 > 0$ and $u_1 = v_1 \leq w_1$ and B be a point with coordinates (u_2, v_2, w_2, t_2) such that $u_2, v_2, w_2, t_2 > 0$ and $v_2 = w_2 \leq u_2$. Consider a finite set of linear inequalities of the form $g_i \geq 0$, $i = 1, \dots, m$ such that for all $i \in \{1, \dots, m\}$, $g_i(A) \geq 0$ and $g_i(B) \geq 0$. Moreover suppose that, for each given i, only one of u, v, w appears in g_i, which means that the plane $g_i = 0$ passes at least through two of the points P, Q, R.*

Then there exists a point D with coordinates $(\lambda/3, \lambda/3, \lambda/3, 1-\lambda)$, $0 \leq \lambda \leq 1$ such $g_i(D) \geq 0$ for all $i \in \{1, \dots, m\}$.



Proof : Let I be the point with coordinates $(1/3, 1/3, 1/3, 0)$. Without loss of generality, we may assume that, for all i, the plane $\{g_i = 0\}$ intersects the segment IS in a point M_i between I and S.

Let i be such that $g_i(R) = 0$. Then the intersection of $\{g_i = 0\}$ with the triangle RIS is the segment RM_i.

Let A_1 and A_2 be the points of IS defined by

$$A_1 S = \bigcup_{\substack{RM_i \text{ lies above } A \\ i\in\{1,\dots,m\}}} M_i S$$

and

$$IA_2 = \bigcup_{\substack{RM_i \text{ lies below } A \\ i\in\{1,\dots,m\}}} IM_i$$

respectively.

Similarly, for all i such that $g_i(P) = 0$ we define $B_1, B_2 \in IS$ such that

$$B_1 S = \bigcup_{\substack{PM_i \text{ lies above } B \\ i\in\{1,\dots,m\}}} M_i S$$

and

$$IB_2 = \bigcup_{\substack{PM_i \text{ lies below } B \\ i\in\{1,\dots,m\}}} IM_i$$

respectively.

Let B' be the point of intersection of PB with IS. We will prove that A_1 belongs to the interval $[SB']$ and B' to $[SA_2]$. And then put $D = B'$. This will complete the proof.

Let us show that $B' \in [SA_2]$. Now, if A_2 is defined by a plane of the form $g_i = 0$ with $g_i(R) = 0$ and $g_i(P) = 0$, then necessarily, B' is closer to S than A_2.

So we may assume that A_2 is defined by a plane of the form $u - kt = 0$.

If A_2 is closer to S than B', then P and B' are both in the same half-space whose boundary is the plane A_2QR with equation $u - kt = 0$. Then $(u-kt)(B) > 0$. But, because, $(u-kt)(A) < 0$ so also $(u-kt)(B) < 0$, which is a contradiction. We prove the same way that $A_1 \in [SB']$. So B' lies between A_1 and A_2 as desired. □

Applying the Tetrahedron Lemma to all the relations of the form (65) or (66) ensures the existence of a point D on IS satisfying the same relations. Consider a point $\delta \in \operatorname{Sper} A$ such that ν_δ has the same three approximate roots Q_4, Q_5, Q_6 and satisfying moreover the fact that the coordinates $(\nu(u_1), \nu(Q'_4), \nu(Q'_5), \nu(Q'_6))$ correspond to the point D, then $\delta \in C$. So we are reduced to the limit case (see Remark (6.7)) where all the approximate roots are pairwise comparable for the couple (α, δ) and for the couple (β, δ). This defines two connected sets C_1 and C_2, avoiding all $\{g_i = 0\}$, and such that C_1 contains α and δ and C_2 contains β and δ. So letting $C = C_1 \cup C_2$ gives a connected set as required.

This settles the last remaining case ((Q_4, Q_5, Q_6) pairwise incomparable) and with it Theorem 1.14.

References

[1] S. Abhyankar and T.T. Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation I.* Reine Agew. Math. 260, 47–83 (1973).

[2] S. Abhyankar and T.T. Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation II.* Reine Agew. Math. 261, 29–54 (1973).

[3] C. Andradas, L. Brücker, J.M. Ruiz, *Constructible Sets in Real Geometry*, Springer (1996).

[4] D. Alvis, B. Johnston, J.J. Madden, *Local structure of the real spectrum of a surface, infinitely near points and separating ideals.* Preprint.

[5] R. Baer, Uber nicht-archimedisch geordnete Körper (Beitrage zur Algebra). Sitz. Ber. Der Heidelberger Akademie, 8 Abhandl. (1927).

[6] G. Birkhoff and R. Pierce, *Lattice-ordered rings.* Annales Acad. Brasil Ciênc. 28, 41–69 (1956).

[7] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle.* Springer–Verlag, Berlin 1987.

[8] S.D. Cutkosky, B. Teissier, *Semi-groups of valuations on local rings*, Michigan Mat. J., Vol. 57, pp. 173-193 (2008).

[9] C. N. Delzell, *On the Pierce–Birkhoff conjecture over ordered fields*, Rocky Mountain J. Math.

[10] L. Fuchs, *Telweise geordnete algebraische Strukturen.* Vandenhoeck and Ruprecht, 1966.

[11] R. Goldin and B. Teissier, *Resolving singularities of plane analytic branches with one toric morphism.*, Resolution of singularities (Obergurgl, 1997), 315–340, Progr. Math., 181, Birkhauser, Basel, 2000.

[12] Herrera Govantes, F. J.; Olalla Acosta, M. A.; Spivakovsky, M. *Valuations in algebraic field extensions*, Journal of Algebra , Vol. 312 , N. 2 , pp. 1033–1074 (2007).

[13] F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky, B. Teissier, *Extending a valuation centered in a local domain to the formal completion.*, Proc. London Math. Soc., first published online March 23, 2012 doi:10.1112/plms/pds002

[14] M. Henriksen and J. Isbell, *Lattice-ordered rings and function rings.* Pacific J. Math. 11, 533–566 (1962).

[15] I. Kaplansky, *Maximal fields with valuations I.* Duke Math. J., 9:303–321 (1942).

[16] I. Kaplansky, *Maximal fields with valuations II.* Duke Math. J., 12:243–248 (1945).

[17] W. Krull, Allgemeine Bewertungstheorie, J. Reine Angew. Math. 167, 160–196 (1932).

[18] T.C. Kuo, *Generalized Newton–Puiseux theory and Hensel's lemma in* $\mathbb{C}[[x, y]]$. Canadian J. Math., (6) XLI, 1101–1116 (1989).

[19] T.C. Kuo, *A simple algorithm for deciding primes in* $\mathbb{C}[[x, y]]$. Canadian J. Math., 47 (4), 801–816 (1995).

[20] M. Lejeune-Jalabert, *Thèse d'Etat.* Université Paris 7 (1973).

[21] F. Lucas, J.J. Madden, D. Schaub and M. Spivakovsky, *On connectedness of sets in the real spectra of polynomial rings*, Manuscripta Math. 128, 505-547, 2009.

[22] F. Lucas, D. Schaub and M. Spivakovsky, *Approximate roots of a valuation and the Pierce-Birkhoff Conjecture*, Ann. Fac. Sci. Toulouse, Mathématique, Série 6, Vol. XXI, Fasc. 2, 259-342, 2012.

[23] S. MacLane, *A construction for prime ideals as absolute values of an algebraic field.* Duke Math. J. 2, 492–510 (1936).

[24] S. MacLane, *A construction for absolute values in polynomial rings.* Transactions of the AMS 40, 363–395 (1936).

[25] S. MacLane and O.F.G. Schilling, *Zero-dimensional branches of rank one on algebraic varieties.* Ann. of Math. 40, 3 (1939).

[26] J. J. Madden, *Pierce–Birkhoff rings.* Arch. Math. 53, 565–570 (1989).

[27] J. J. Madden, The Pierce-Birkhoff conjectures (unpublished preprint).

[28] L. Mahé, *On the Pierce–Birkhoff conjecture.* Rocky Mountain J. Math. 14, 983–985 (1984).

[29] L. Mahé, *On the Pierce–Birkhoff conjecture in three variables.*, J. Pure Appl. Algebra, 211, 459-470 (2007).

[30] M. Marshall, *Orderings and real places of commutative rings.* J. Alg. 140, 484–501 (1991).

[31] M. Marshall, *The Pierce–Birkhoff conjecture for curves.* Can. J. Math. 44, 1262–1271 (1992).

[32] H. Matsumura, *Commutative Algebra.* Benjamin/Cummings Publishing Co., Reading, Mass., 1970.

[33] A. Prestel, *Lectures on formally real fields*, Lecture Notes in Math., Springer–Verlag–Berlin, Heidelberg, New York, 1984.

[34] A.Prestel, C.N.Delzell *Positive Polynomials*, Springer monographs in mathematics, 2001.

[35] S. Priess-Crampe, *Angeordnete strukturen: gruppen, körper, projektive Ebenen*, Springer–Verlag–Berlin, Heidelberg, New York, 1983.

[36] N. Schwartz, *Real closed spaces.* Habilitationsschrift, München 1984.

[37] M. Spivakovsky, *Valuations in function fields of surfaces.* Amer. J. Math 112, 1, 107–156 (1990).

[38] M. Spivakovsky, *A solution to Hironaka's polyhedra game.* Arithmetic and Geometry, Vol II, Papers dedicated to I. R. Shafarevich on the occasion of his sixtieth birthday, M. Artin and J. Tate, editors, Birkhäuser, 1983, pp. 419–432.

[39] B. Teissier, *Valuations, deformations and toric geometry.*, Proceedings of the Saskatoon Conference and Workshop on valuation theory, Vol. II, F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, 33 (2003), 361-459.

[40] M. Vaquié, *Famille admise associée à une valuation de* $K[x]$, Séminaires et Congrès 10, edited by Jean-Paul Brasselet and Tatsuo Suwa, 2005, pp. 391–428.

[41] M. Vaquié, *Extension d'une valuation*, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3439–3481.

[42] M. Vaquié, *Algèbre graduée associée à une valuation de* $K[x]$, Adv. Stud. Pure Math., 46, Math. Soc. Japan, Tokyo, 2007.

[43] M. Vaquié, *Famille admissible de valuations et défaut d'une extension*, J. Algebra 311 (2007), no. 2, 859–876.

[44] M. Vaquié, *Valuations*, 539–590, Progr. Math., 181, Birkhäuser, Basel, 2000.

[45] S. Wagner, *On the Pierce-Birkhoff Conjecture for Smooth Affine Surfaces over Real Closed Fields*, Ann. Fac. Sci. Toulouse, Math. (6) 19, Spec. Issue, 221-242 (2010).

[46] O. Zariski, P. Samuel *Commutative Algebra*,Vol. II, Springer Verlag.